Entanglement spectroscopy of SU(2) broken phases in 2D

Vincenzo Alba[1]

[1]Ludwig-Maximilians Universität, München

King's College London, June 4, 2014

Acknowledgements

Fabian Kolley
LMU

Stefan Depenbrock
Irvine

Ian P. McCulloch
University of Queensland

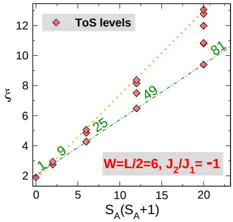
Uli Schollwöck
LMU Munich

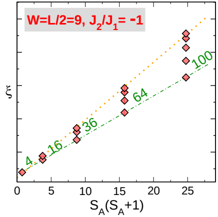
[F. Kolley, S. Depenbrock, I. P. McCulloch, U. Schollwöck, V. A., PRB **88**, 144426]

Outline

- Introduction: entanglement.
- Entanglement spectrum (ES).
- Entanglement spectrum in **2D SU(2)-broken** phases.
- **Ground state** ES of the **2D** J_1-J_2 Heisenberg model:
  - Triangular & kagomé lattice.
- Low-lying ES levels ⇔ SU(2) **towers of states**.
- ES (combined with **DMRG**) as a tool to detect **continuous** symmetry broken phases.

  [F. Kolley, S. Depenbrock, I. P. McCulloch, U. Schollwöck, V. A., PRB **88**, 144426]
- Bonus: entanglement in multipartite systems (**negativity**).





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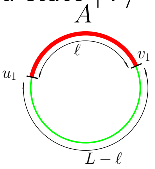
Entanglement

- Consider a quantum system in d dimensions in the ground state $|\Psi\rangle$

$$\rho \equiv |\Psi\rangle\langle\Psi|$$



- If the system is **bipartite**:

$$H = H_A \otimes H_B \rightarrow \rho_A = Tr_B \rho$$

- How to quantify the entanglement (**quantum** correlations) between A and B?
  - von Neumann entropy $S_A = -Tr \rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i$

- **1D conformal** invariant systems [Holzhey,Larsen,Wilczek,1994] [Calabrese, Cardy,2004]

$$\boxed{S_A = \frac{c}{3} \log \ell}$$

c central charge

- **2D topologically** ordered phases [Kitaev,Preskill,2006][Levin,Wen, 2006]

$$\boxed{S_A = \alpha \ell^{d-1} + \gamma}$$

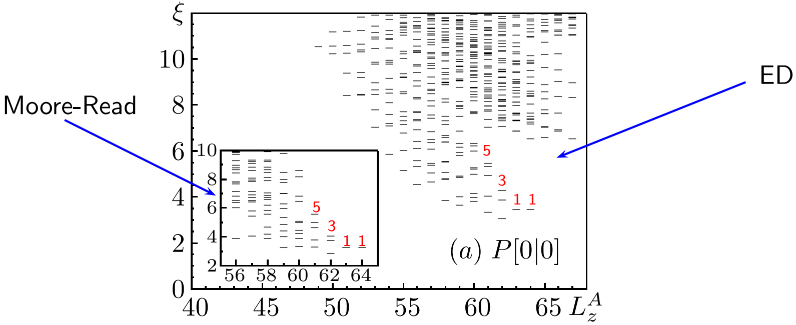
γ **topological entropy**

Entanglement spectrum (ES)

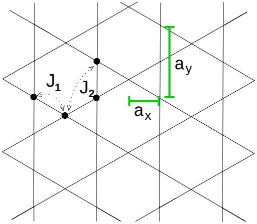
- Given the reduced density matrix ρ_A:

$$\{\lambda\} = \sigma(\rho_A) \quad \Rightarrow \quad \rho_A = e^{-\mathcal{H}_A} \quad \Rightarrow \quad \sigma(\mathcal{H}_A) \equiv -\log(\{\lambda\})$$

- Fractional Quantum Hall: the ES retains features of the critical edge modes (ES$\rightarrow$ edge spectra correspondence) [Li,Haldane, 2008]



- More general: the relevant information to describe the physics of a system is encoded in the "low energy" part of entanglement spectra.



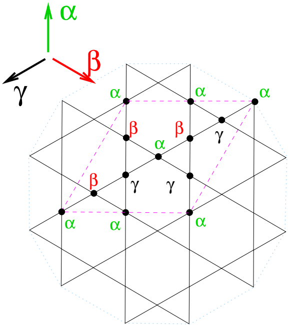
$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle ij \rangle\rangle} S_i \cdot S_j$$

$J_1 > 0$

[Yan,Huse,White,2011]

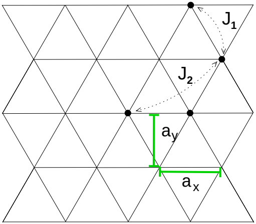
[Depenbrock, McCulloch, Schollwöck,2012]

[Jiang,Wang,Balents,2013]



- At large **ferromagnetic** $J_2 < 0 \Rightarrow$ SU(2) is broken (here $J_2/J_1 = -1$).
- **Coplanar** magnetic order ($\sqrt{3} \times \sqrt{3}$ state).
- **Complete** breaking of SU(2) in the thermodynamic limit.
- **Three** ferromagnetic sublattices.

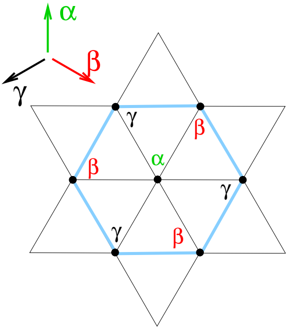
J_1-J_2 Heisenberg model on the triangular lattice



$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle ij \rangle\rangle} S_i \cdot S_j \qquad J_1 > 0$$

[Bernu,Lhuillier,Pierre,1992]

[White,Chernyshev,2007]



- At large **ferromagnetic** $J_2 < 0 \Rightarrow$ SU(2) is broken (here $J_2/J_1 = -1$).
- **Coplanar** magnetic order (120° state).
- **Complete** breaking of SU(2) in the thermodynamic limit.
- **Three** ferromagnetic sublattices.

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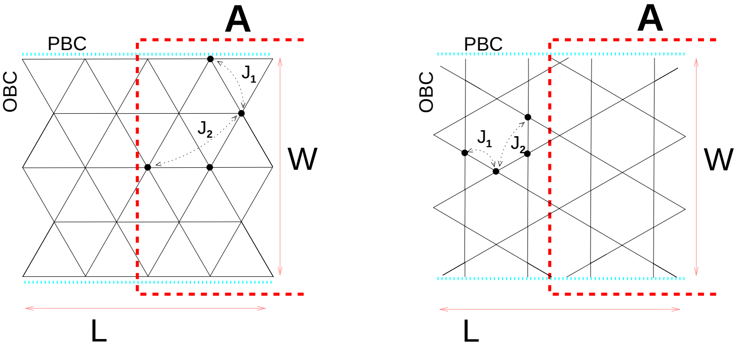
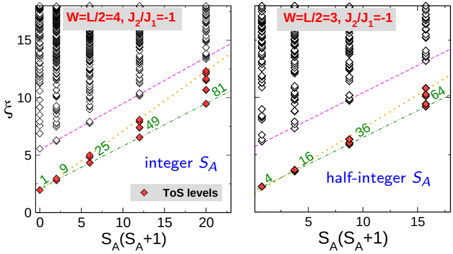
- Non abelian **SU(2) symmetric DMRG** simulations.
- Cylindrical geometry.
- Half-cylinder ES ($W = L/2$).

[S. White,1992]
[U. Schollwöck,2005,2011]
[E. M. Stoudenmire,2012]

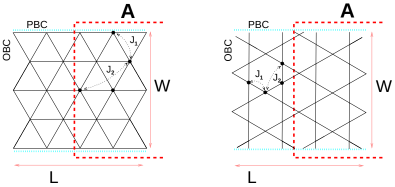
$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle ij \rangle\rangle} S_i \cdot S_j$$



Kagomé cylinders



S_A subsystem total spin



- ES levels form SU(2) **multiplets**.
- Prominent **tower** structures in **low-lying** levels.
- **Linear** behavior with $\mathbf{S_A(S_A+1)}$.
- Entanglement **gap**.
- Tower **multiplicities** as $\mathbf{(2S_A+1)^2}$.

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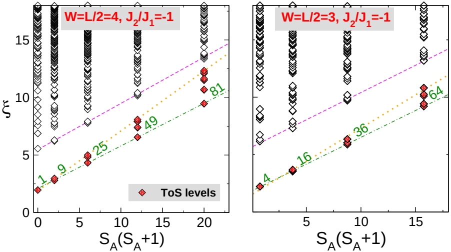
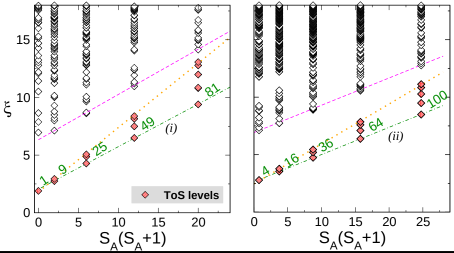
ES in SU(2) broken phases: DMRG results

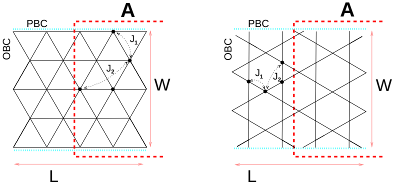
Kagomé cylinders



Triangular cylinders



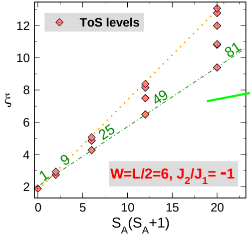
S_A subsystem total spin



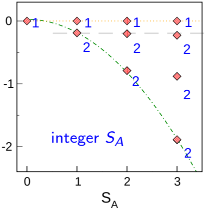
- ES levels form SU(2) **multiplets**.
- Prominent **tower** structures in **low-lying** levels.
- **Linear** behavior with $\mathbf{S_A(S_A+1)}$.
- Entanglement **gap**.
- Tower **multiplicities** as $(\mathbf{2S_A+1})^2$.
- Same structure for both lattices.

Tower of states substructure

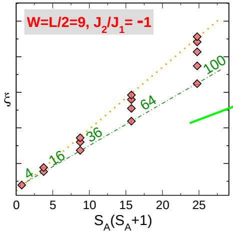
Triangular lattice ES



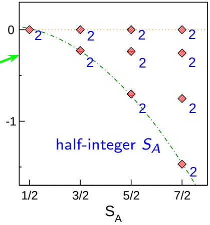
shifted towers



“lines” of ES levels



shifted towers



full pairing

- **Pairs** of **degenerate** SU(2) multiplets.
- Regular distribution of levels.

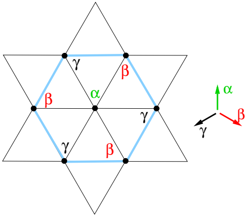
Tower of states mechanism in energy spectra

$$\mathcal{H} = J_1 \sum_{\langle ij \rangle} S_i \cdot S_j + J_2 \sum_{\langle\langle ij \rangle\rangle} S_i \cdot S_j$$

- At $J_2/J_1 = -1 \Rightarrow$ **magnetic order** $\Rightarrow SU(2)$ **(continuous) symmetry breaking**.
- Problem: the system is in a **finite volume** V (no symmetry breaking).
- Symmetry breaking and finite size energy spectra: **Anderson tower of states** [Anderson,52][Fisher,89][Bernu et al.,92]

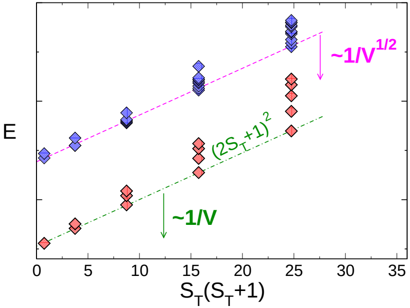
$$\mathcal{H} \approx \underbrace{\frac{(S_T^2 - S_\alpha^2 - S_\beta^2 - S_\gamma^2)}{2\chi_\perp V} + \frac{(S^{z'})^2}{2V}\Big(\frac{1}{\chi_\perp} - \frac{1}{\chi_\parallel}\Big)}_{\mathcal{H}_{ToS}} + \mathcal{H}_{sw}$$

- S_T total lattice spin
- $S_{\alpha,\beta,\gamma}$ total sublattice spin
- $\chi_\perp, \chi_\parallel$ spin susceptibilities
- $S^{z'} \in [-S_T, S_T]$ effective spin variable
- $\mathcal{H}_{sw}$ spin wave Hamiltonian



$$(S_T^2 - S_\alpha^2 - S_\beta^2 - S_\gamma^2) = 2(S_\alpha \cdot S_\beta + S_\alpha \cdot S_\gamma + S_\beta \cdot S_\gamma)$$

SU(2) tower of states energy spectrum



- Asymptotic **gap** in the spectrum.
- Linear behavior with $S_T(S_T+1)$.
- ToS **multiplicity** as $(2S_T+1)^2$.
- ToS **degeneracy** is lifted as $\sim (S_T^{z'})^2$.

$$\mathcal{H} \approx \frac{(S_T^2 - S_\alpha^2 - S_\beta^2 - S_\gamma^2)}{2\chi_\perp V} + \frac{(S^{z'})^2}{2V}\left(\frac{1}{\chi_\perp} - \frac{1}{\chi_\parallel}\right) + \mathcal{H}_{sw}$$

$$\Delta\mathcal{H}_{sw} \sim 1/V^{1/2}$$

- **ToS spectroscopy** routinely used to detect SU(2) broken phases with exact diagonalization [Bernu,Lhuillier,Pierre,1992]

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Tower of states structure and ES

- The lower part of the **ground state** ES in systems with **continuous** symmetry breaking is described by the **tower of states** Hamiltonian.

[Grover,Metlitski,(2011)]

[V.A.,Haque,Läuchli, (2013)]

$$\boxed{\mathcal{H}_E \approx \mathcal{H}^{(A)}_{Tos}/T_E} \qquad T_E \sim v_s/V^{1/2}$$

- Heisenberg J_1-J_2 model $\Rightarrow$ Effective **entanglement Hamiltonian**

$$\mathcal{H}_E \approx \frac{S_A(S_A+1)}{v_s\chi_\perp W} - \frac{(S_A^{z'})^2}{v_s W}\left(\frac{1}{\chi_\perp} - \frac{1}{\chi_\|}\right) \qquad V^{1/2} \sim W$$

  - Tower collapse as $\sim 1/W$.
  - Linear behavior as $S_A(S_A+1)$.
  - Tower substructure given by $(S_A^{z'})^2$.

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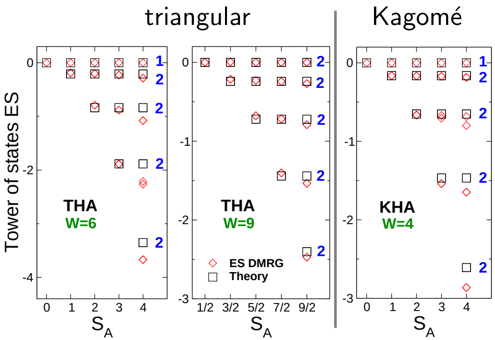
Tower of states structure and the ES: numerical check



- A **one parameter** fit reproduces the behavior of the entire ToS structure.

$$\alpha(S_A^{z'})^2$$

- $S_A^{z'}$ and $-S_A^{z'}$ are **degenerate** $\Rightarrow$ SU(2) multiplets **pairs**

$$\mathcal{H}_E \approx \cancel{\frac{S_A(S_A+1)}{v_s\chi_\perp W}} - \frac{(S_A^{z'})^2}{v_s W}\left(\frac{1}{\chi_\perp} - \frac{1}{\chi_\parallel}\right)$$

$$S_A^{z'} = -S_A, -S_A+1, \dots, S_A-1, S_A$$

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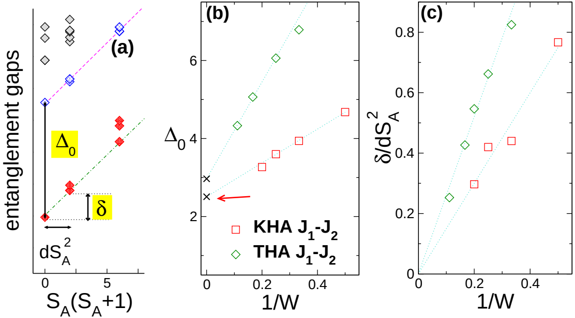
Tower of states finite size behavior: entanglement gap



$$\mathcal{H}_E \approx \frac{S_A(S_A+1)}{v_s \chi_\perp W} - \frac{(S_A^{z'})^2}{v_s W}\Big(\frac{1}{\chi_\perp} - \frac{1}{\chi_\parallel}\Big)$$

- δ is vanishing as $1/V^{1/2} \sim 1/W$ with increasing V.
- The entanglement gap Δ_0 remains "finite".

U(1) broken phases: the 2D Bose-Hubbard model

- Similar analysis for the 2D Bose-Hubbard model in the superfluid phase.

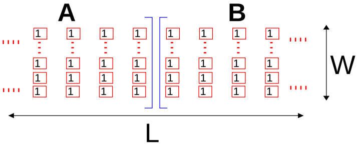
$$\mathcal{H} = -\sum_{\langle ij\rangle}(b_i^\dagger b_j + h.c.) + \frac{U}{2}\sum_i n_i(n_i - 1)$$

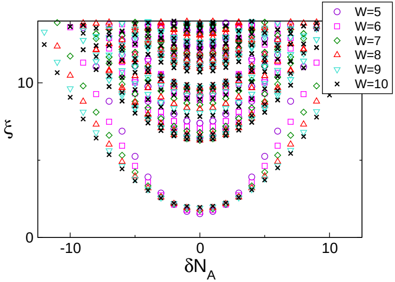
- **Superfluid** phase $\Rightarrow$ **U(1)** symmetry breaking.
- Tower of states Hamiltonian given as:

$$\mathcal{H}_{ToS} \sim \frac{(\delta N)^2}{V}$$

- **Parabolic** low-lying structure in the entanglement spectrum

$$\mathcal{H}_E \sim \frac{(\delta N_A)^2}{W}$$





[V.A,Haque,Läuchli,2013]

Summary

- In **SU(2)-broken** phases the ground state **ES** reflects the **low-energy** physics of the system.
- Low-lying ES levels are in correspondence with the **tower of states** energy spectrum.
- ToS structures in ES are divided from the rest by an **entanglement gap**.
- **Entanglement spectroscopy** as a **tool** to detect symmetry broken phases with **DMRG**.
- Only **ground state** calculations needed (as opposed to conventional energy tower of states spectroscopy)

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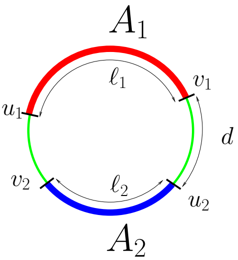
Entanglement of multi-intervals

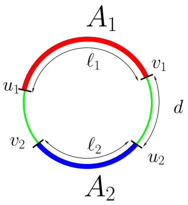
- Single interval entanglement $\Rightarrow$ central charge.
- Much more **universal** information is contained in the entanglement between **many** disjoint intervals.

Two disjoint intervals



- How can we measure the **mutual** entanglement between A_1 and A_2?

The mutual information



- Renyi **mutual information** between A_1 and A_2:

$n \in \mathbb{N}$

$$I^{(n)}[A_1 : A_2] \equiv S^{(n)}_{A_1} + S^{(n)}_{A_2} - S^{(n)}_{A_1 \cup A_2}$$

Renyi entropies

$$S^{(n)}_A \equiv -\frac{1}{n-1} \log \mathrm{Tr} \rho^n_A$$

- The mutual information does **not** measure the quantum **entanglement** between A_1 and A_2. [Plenio, Virmani, 2007]

- $I^{(n)}[A_1 : A_2]$ provides only an **upper bound** on the entanglement between two intervals.

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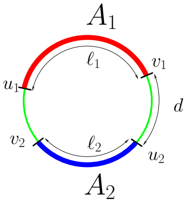
Logarithmic negativity

- The logarithmic **negativity** is a computable measure of entanglement between two intervals.

$$\boxed{\mathcal{E} \equiv \log \mathrm{Tr}|\rho_{A_1\cup A_2}^{T_2}|} \qquad \mathrm{Tr}|\rho_{A_1\cup A_2}| \equiv \sum_i |\lambda_i| = \sum_{\lambda_i>0} \lambda_i - \sum_{\lambda_i<0} \lambda_i$$

[Vidal-Werner,2002]



- Trick/Cost: **partial transposition** with respect to degrees of freedom of A_2.

$$\langle e_i^{(1)} e_j^{(2)} | \boxed{\rho_{A_1\cup A_2}^{T_2}} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho_{A_1\cup A_2} | e_k^{(1)} e_j^{(2)} \rangle$$

- $\mathcal{E}$ measures "how much" the eigenvalues of ρ^{T_2} are negative.

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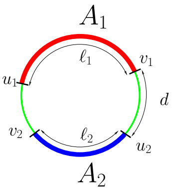
Scaling behavior: CFT results

- $\mathrm{Tr}(\rho^{T_2}_{A_1\cup A_2})^n$ can be computed in CFT via **replica** trick.



- Two **adjacent** intervals ($u_2 \to v_1$):

[Calabrese et al., 2012,2013]

$$\boxed{\mathcal{E} = \frac{c}{4}\log\frac{\ell_1\ell_2}{\ell_1+\ell_2} + \mathrm{cnst}}$$

- Two **disjoint** intervals: only $\mathrm{Tr}(\rho^{T_2}_{A_1\cup A_2})^n$ are known

$$\mathrm{Tr}(\rho^{T_2}_{A_1\cup A_2})^n = c_n[\ell_1\ell_2(1-y)]^{-\frac{c}{6}(n-\frac{1}{n})}\boxed{G_n(y)}$$

- $G_n(y)$ is a **universal** scaling function of the four point ratio y

$$y \equiv \frac{|u_1-v_1||u_2-v_2|}{|u_1-u_2||v_1-v_2|}$$

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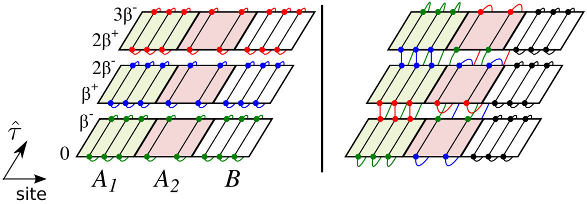
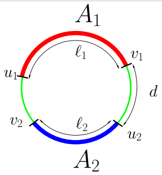
Entanglement negativity and (Q)Monte Carlo simulations

- $\mathrm{Tr}(\rho_{A_1\cup A_2}^{T_2})^n$ can be obtained in (Quantum) **Monte Carlo**.
- **Replica trick** representation.



[V.A., J. Stat. Mech. P05013 (2013)]
[Chung, V.A., Bonnes, Chen, Läuchli, arXiv:1312.1168]

Numerical checks



- 1D hard-core bosons.

$$\mathcal{H} = -t \sum_i (b_i^\dagger b_{i+1} + h.c.)$$

- Consider the **scale invariant** ratios R_n:

$$R_n \equiv \frac{\mathrm{Tr}(\rho_{A_1 \cup A_2}^{T_2})^n}{\mathrm{Tr}\rho_{A_1 \cup A_2}^n}$$

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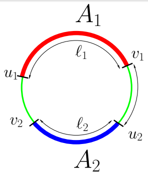
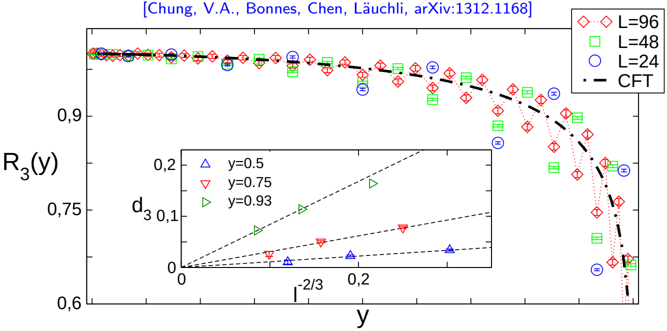


[Chung, V.A., Bonnes, Chen, Läuchli, arXiv:1312.1168]



$$d_3 \equiv R_3 - R_3^{CFT}$$

$$R_3 = R_3^{CFT} + \ell^{-2/3} g^{(o/e)}(y) + \dots$$

Summary

- A lot of **universal** information about critical systems can be extracted from the entanglement between two **disjoint** intervals.
- Entanglement **negativity**.
- Negativity-related quantities can be effectively calculated in **(Q) Monte Carlo**.

Thanks!