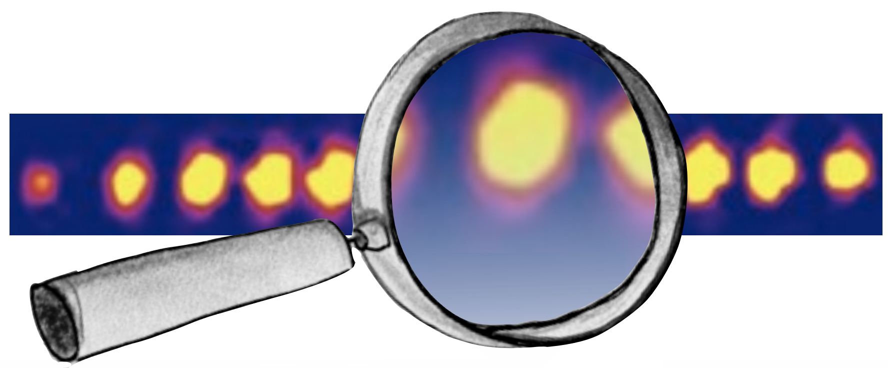
Efficient State Tomography with Matrix Product Structures



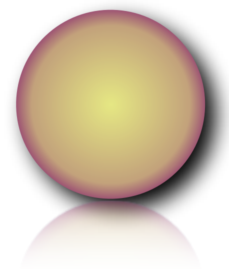
T. Baumgratz

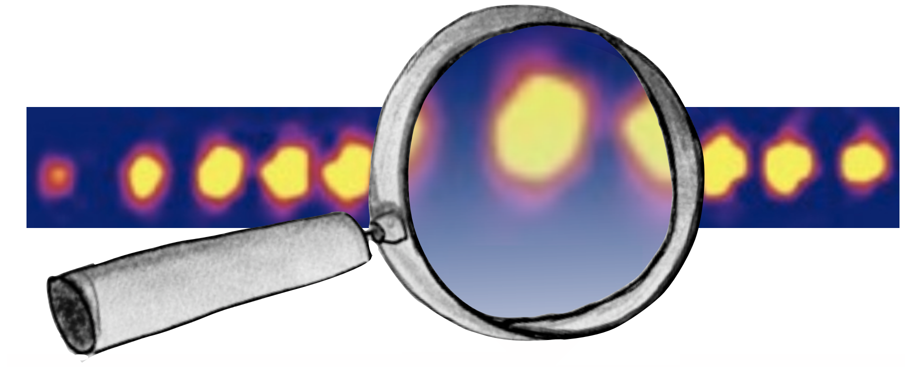
ulm university universität ulm

Joint work with
D. Gross, M. Cramer, and M.B. Plenio

London, 2nd of June 2014

Efficient State Tomography





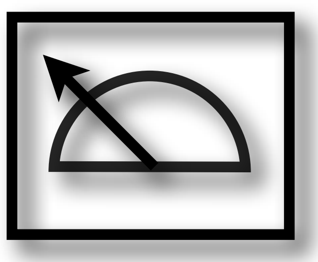
Measurements:

$$p_k = \mathrm{tr}\left[\hat{\Pi}_k \hat{\varrho}\right]$$

→

Reconstruct density matrix:

$$\hat{\varrho} = \frac{\mathbb{1}}{2} + \sum_{i=1}^{3} \frac{\langle \hat{\sigma}_i \rangle}{2}\,\hat{\sigma}_i$$

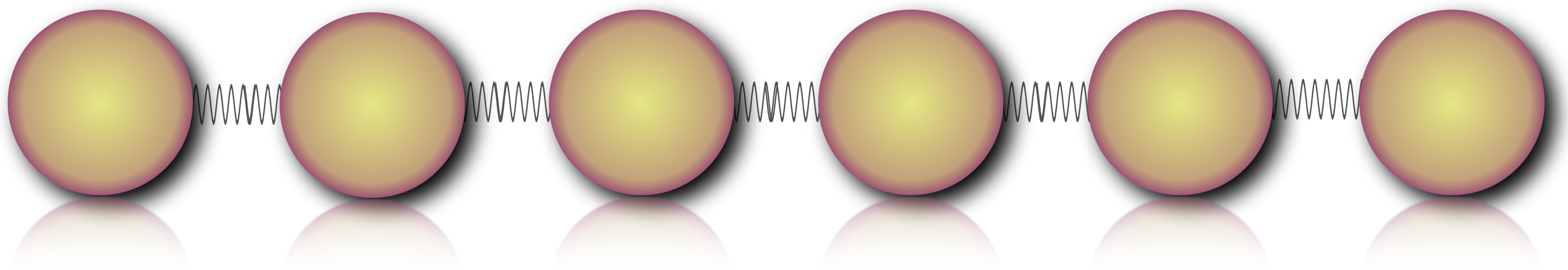


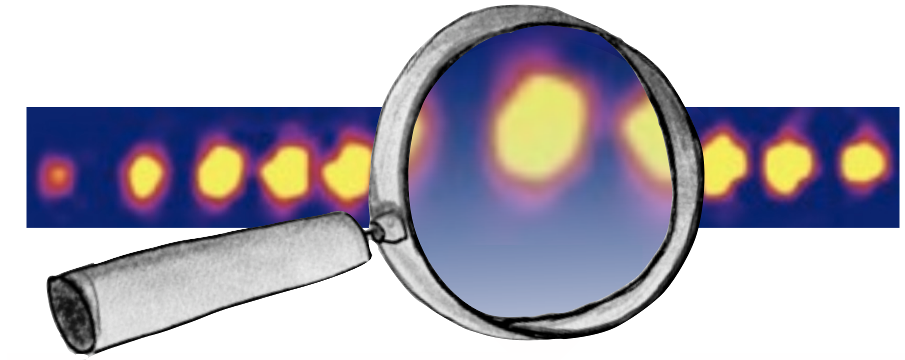
Techniques:

- Linear inversion,
- Maximum Likelihood,
- Mean Bayesian estimation, ...

M. Paris *et al.*, Lecture Notes in Physics **649** (2004).

Efficient State Tomography



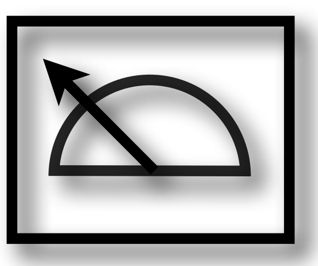


Measurements:

$$p_k = \mathrm{tr}\left[\hat{\Pi}_k \hat{\varrho}\right], \; k = 1, \ldots, d^{2N}$$

Reconstruct density matrix:

$$\hat{\varrho} \in \mathbb{C}^{d^N \times d^N}$$



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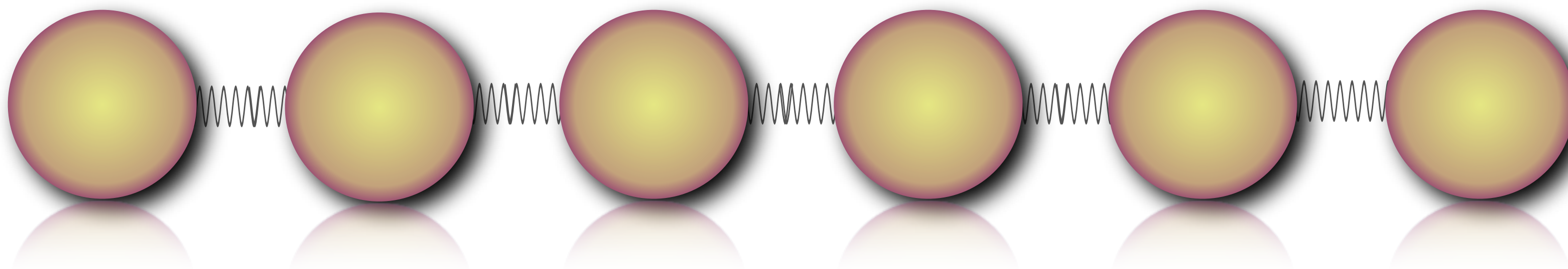
Exponential growth of Hilbert space dimension with the number of subsystems!

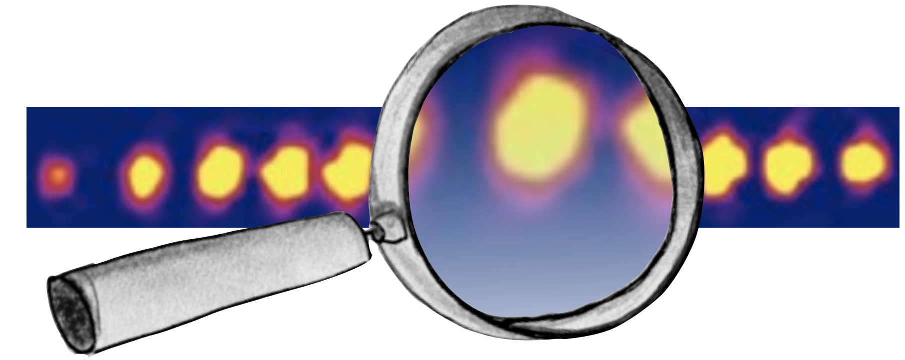
Experiment time

- Problems -

Post-processing resources

Efficient State Tomography





Exponential growth of Hilbert space dimension with the number of subsystems!

- Problems -

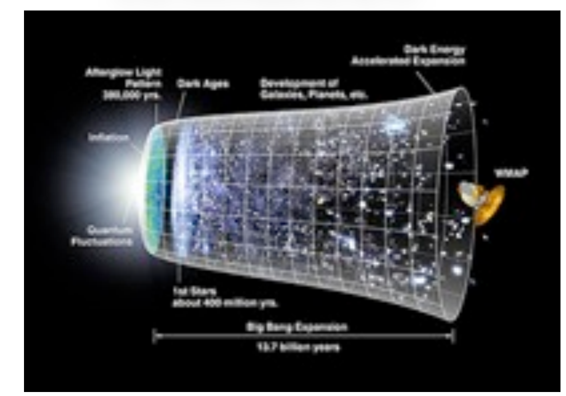
Experiment time, examples:

$N = 8 \;:\; t_{exp} \approx 10\,h$, H. Häffner *et al.*, Nature **438**, 643 (2005).

$N = 14 \;:\; t_{exp} > 300$ days,

$N = 36 \;:\; t_{exp} >$ age of the universe.

Post-processing resources



PRL **106,** 130506 (2011) PHYSICAL REVIEW LETTERS week ending 1 APRIL 2011

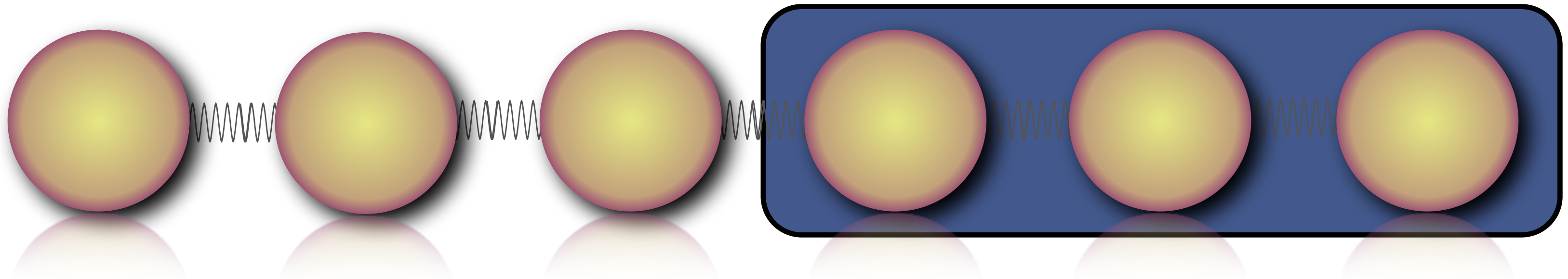
14-Qubit Entanglement: Creation and Coherence

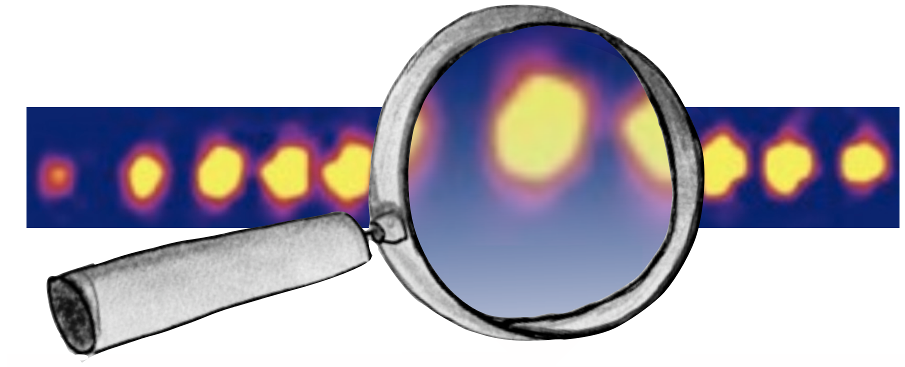
Thomas Monz,[1] Philipp Schindler,[1] Julio T. Barreiro,[1] Michael Chwalla,[1] Daniel Nigg,[1] William A. Coish,[2,3] Maximilian Harlander,[1] Wolfgang Hänsel,[4] Markus Hennrich,[1,*] and Rainer Blatt[1,4]

[1]*Institut für Experimentalphysik, Universität Innsbruck, Technikerstr. 25, A-6020 Innsbruck, Austria*

[2]*Institute for Quantum Computing and Department of Physics and Astronomy, University of Waterloo, Waterloo, ON, N2L 3G1, Canada*

EFFICIENT STATE TOMOGRAPHY





Exponential growth of Hilbert space dimension with the number of subsystems!

- Problems -

Experiment time

Post-processing resources

Reduce the number of measurements.

Choose efficient state representations.

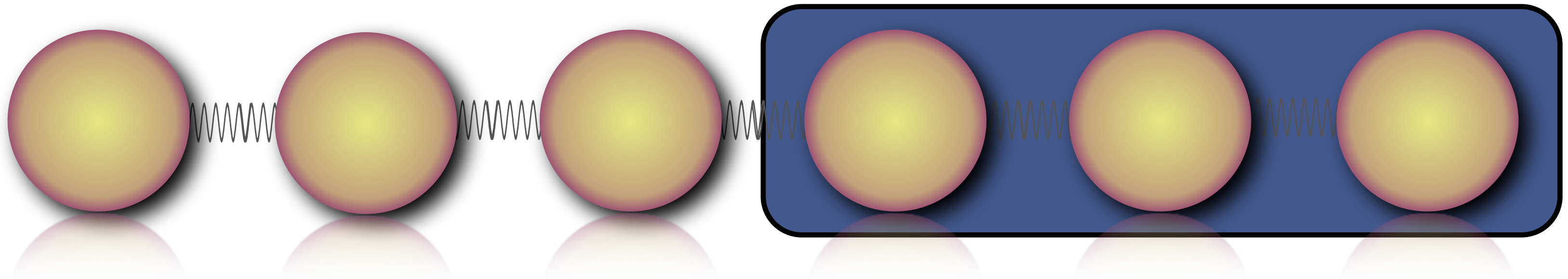
Measure reduced density matrices:

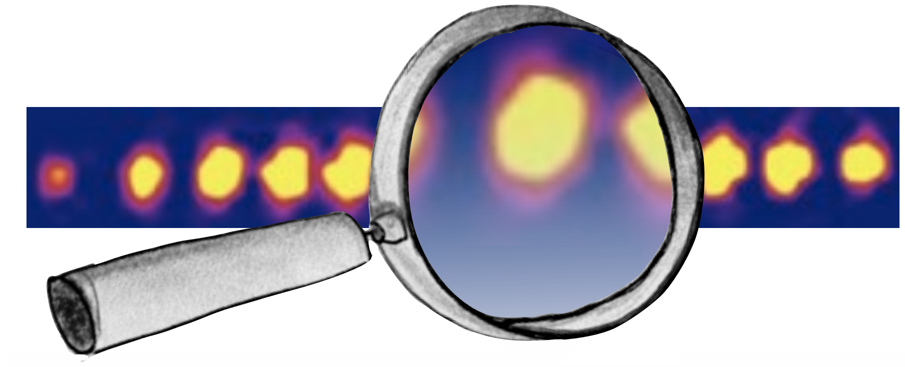
$$\hat{\varrho}_{k,k+1,k+2} = \mathrm{tr}_{\setminus\{k,k+1,k+2\}}\big[\hat{\varrho}\big]$$

Matrix product operator:

$$\hat{\varrho} = \sum_{i_1,\ldots,i_N} B_1[i_1]\cdots B_N[i_N]\; \hat{P}_1^{(i_1)} \otimes \cdots \otimes \hat{P}_N^{(i_N)}$$

EFFICIENT STATE TOMOGRAPHY





Exponential growth of Hilbert space dimension with the number of subsystems!

Experiment time — Problems — Post-processing resources

Reduce the number of measurements.

?

Choose efficient state representations.

Ground, thermal states of local Hamiltonians, W-state, GHZ-state, ...

✓

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$$\hat{\varrho}_{k,k+1,k+2} = \mathrm{tr}_{\setminus\{k,k+1,k+2\}}\left[\hat{\varrho}\right]$$

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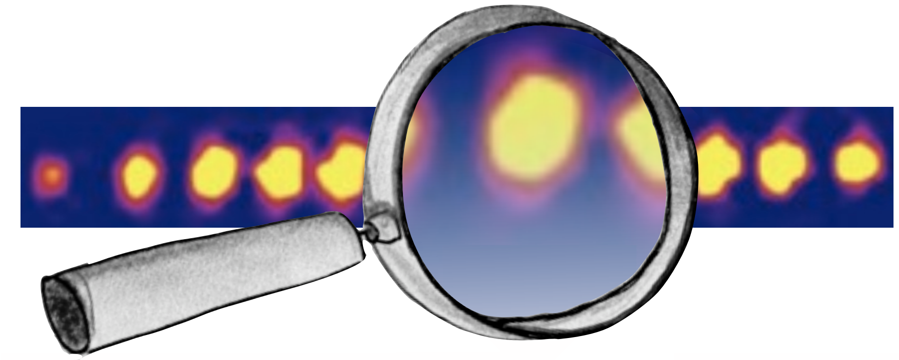
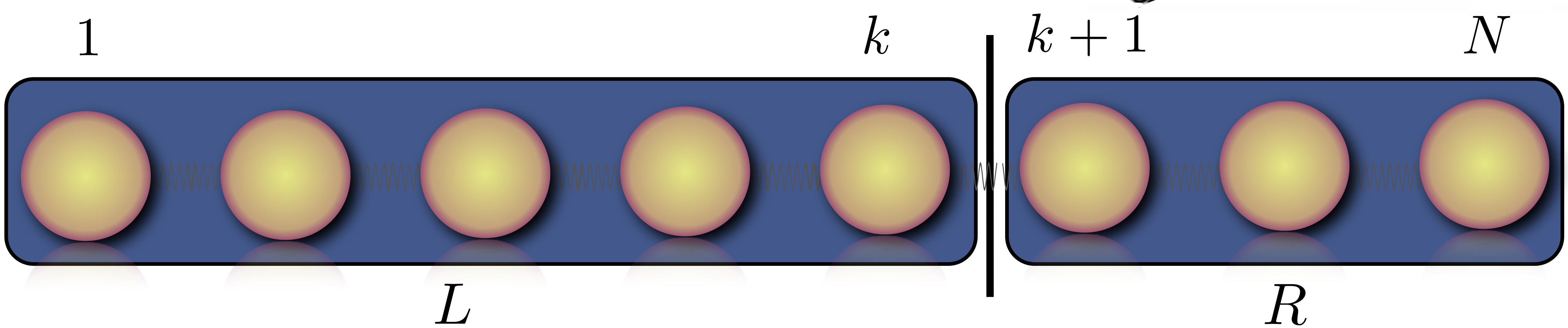
$$\hat{\varrho} = \sum_{i_1,\ldots,i_N} B_1[i_1]\cdots B_N[i_N]\ \hat{P}_1^{(i_1)}\otimes\cdots\otimes\hat{P}_N^{(i_N)}$$

Efficient State Tomography



The Invertibility condition:



Let $l, r \in \mathbb{N}$, $2 \leq l + r \leq N - 2$ and define the linear maps:

$$E_{\{k-l+1,\ldots,k\}}^{\{k+1,\ldots,k+r\}}\left(\hat{X}_{\{k+1,\ldots,k+r\}}\right) = \mathrm{tr}_{\{k+1,\ldots,k+r\}}\left[\hat{\varrho}_{\{k-l+1,\ldots,k+r\}}\hat{X}_{\{k+1,\ldots,k+r\}}\right]$$

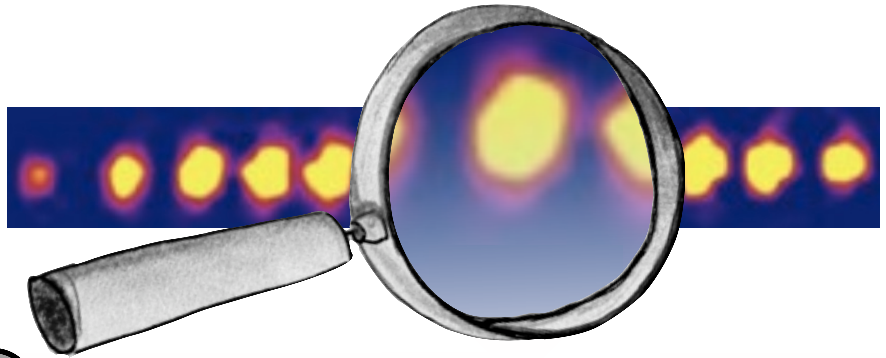
& $$E_{\{1,\ldots,k\}}^{\{k+1,\ldots,N\}}\left(\hat{X}_{\{k+1,\ldots,N\}}\right) = \mathrm{tr}_{\{k+1,\ldots,N\}}\left[\hat{\varrho}\hat{X}_{\{k+1,\ldots,N\}}\right]$$

If with respect to any cut it holds that

$$\mathrm{rank}\left[E_{\{k-l+1,\ldots,k\}}^{\{k+1,\ldots,k+r\}}\right] = \mathrm{rank}\left[E_{\{1,\ldots,k\}}^{\{k+1,\ldots,N\}}\right]$$

than $\hat{\varrho}$ fulfills the invertibility condition.

Efficient State Tomography



Linear growing number of measurements:

Mixed states:

$$\hat{\varrho} = \sum_{i_1,\ldots,i_N=1}^{d^2} B_1[i_1]\cdots B_N[i_N]\,\hat{P}_1^{(i_1)}\otimes\cdots\otimes\hat{P}_N^{(i_N)}$$

Assume that there is a r such that for all $k=0,\ldots,N-r-1$ the sets $\left\{B_{k+1}[i_{k+1}]\cdots B_{k+r}[i_{k+r}]\right\}_{i_{k+1},\ldots,i_{k+r}=1}^{d^2}$ span $\mathbb{C}^{D_{k+1}\times D_{k+r+1}}$.

aka injective MPO.

⇓

$\hat{\varrho}$ satisfies the invertibility condition.

⇓

$\hat{\varrho}$ is uniquely determined by its reduced density matrices on $2r+1$ sites.

Pure states:

D. Pérez-García *et al.*, J. Math. Phys. **47**, 083506 (2006).

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} A_1[i_1]\cdots A_N[i_N]\,|i_1,\ldots,i_N\rangle$$

Assume that there is a r such that for all $k=0,\ldots,N-r-1$ the sets $\left\{A_{k+1}[i_{k+1}]\cdots A_{k+r}[i_{k+r}]\right\}_{i_{k+1},\ldots,i_{k+r}=1}^{d}$ span $\mathbb{C}^{D_{k+1}\times D_{k+r+1}}$.

aka injective MPS.

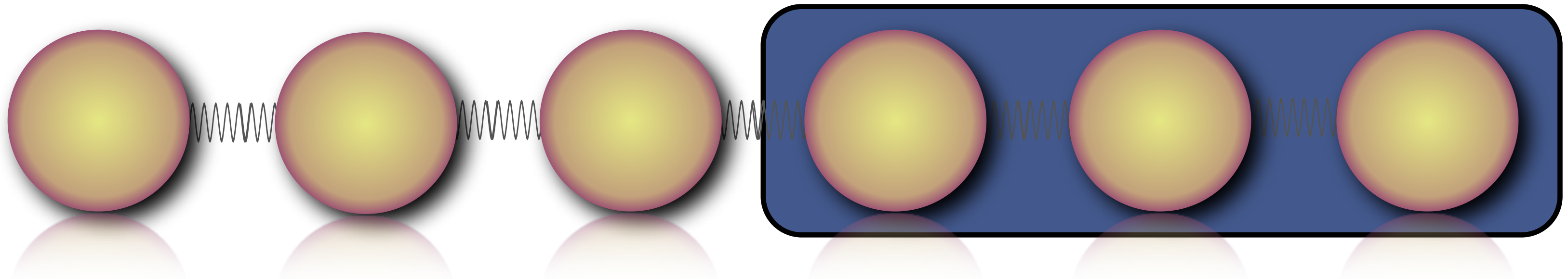
⇓

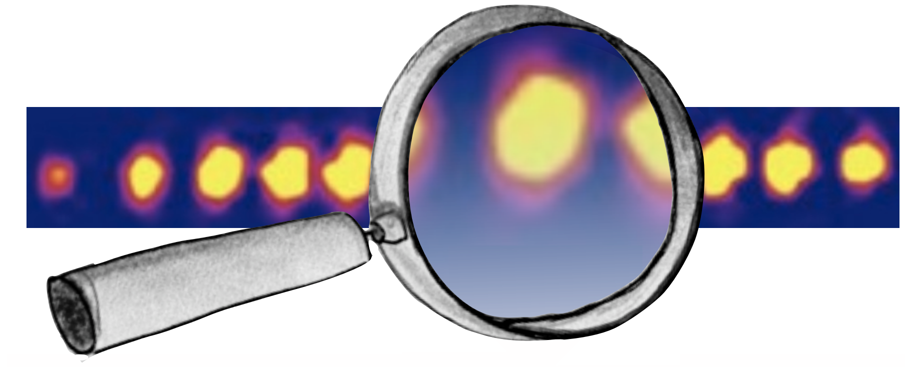
$|\psi\rangle$ is the unique ground state of a local Hamiltonian.

⇓

$|\psi\rangle$ is uniquely determined by its reduced density matrices on $r+1$ sites.

EFFICIENT STATE TOMOGRAPHY





Exponential growth of Hilbert space dimension with the number of subsystems!

Experiment time — Problems — Post-processing resources

Reduce the number of measurements.

✓ Injective matrix product states and operators are uniquely specified locally.

Choose efficient state representations.

Ground, thermal states of local Hamiltonians, W-state, GHZ-state, ... ✓

?

Measure reduced density matrices:

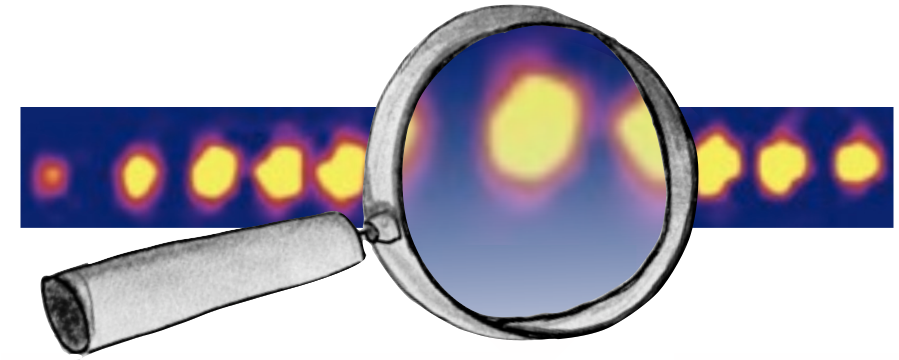
$$\hat{\varrho}_{k,k+1,k+2} = \mathrm{tr}_{\setminus\{k,k+1,k+2\}}\left[\hat{\varrho}\right]$$

→

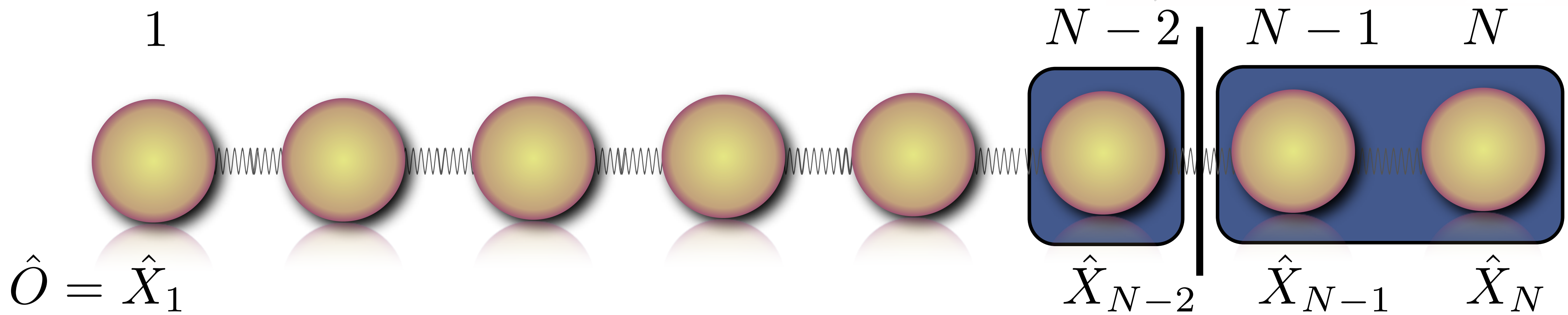
Matrix product operator:

$$\hat{\varrho} = \sum_{i_1,\ldots,i_N} B_1[i_1]\cdots B_N[i_N]\; \hat{P}_1^{(i_1)} \otimes \cdots \otimes \hat{P}_N^{(i_N)}$$

Efficient State Tomography



Compute expectation value: $\mathrm{tr}[\hat{\varrho}\hat{O}]$



Let $\hat{\varrho}$ satisfy the invertibility condition and $l = r = 1$. Then, for all cuts and all $\hat{X}_{k+1} \otimes \hat{X}_{k+2}$ there is a $\hat{Y}_{k+1}$ such that

$$\mathrm{tr}_{k+1}\left[\hat{\varrho}_{k,k+1}\,\hat{Y}_{k+1}\right] = \mathrm{tr}_{k+1,k+2}\left[\hat{\varrho}_{k,k+1,k+2}\,\hat{X}_{k+1} \otimes \hat{X}_{k+2}\right].$$

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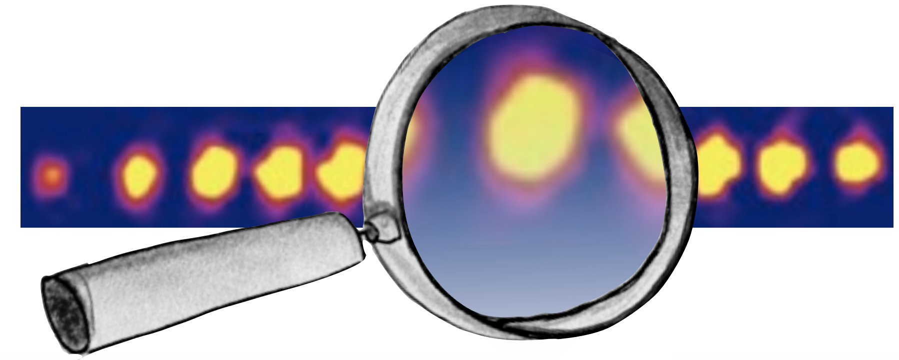

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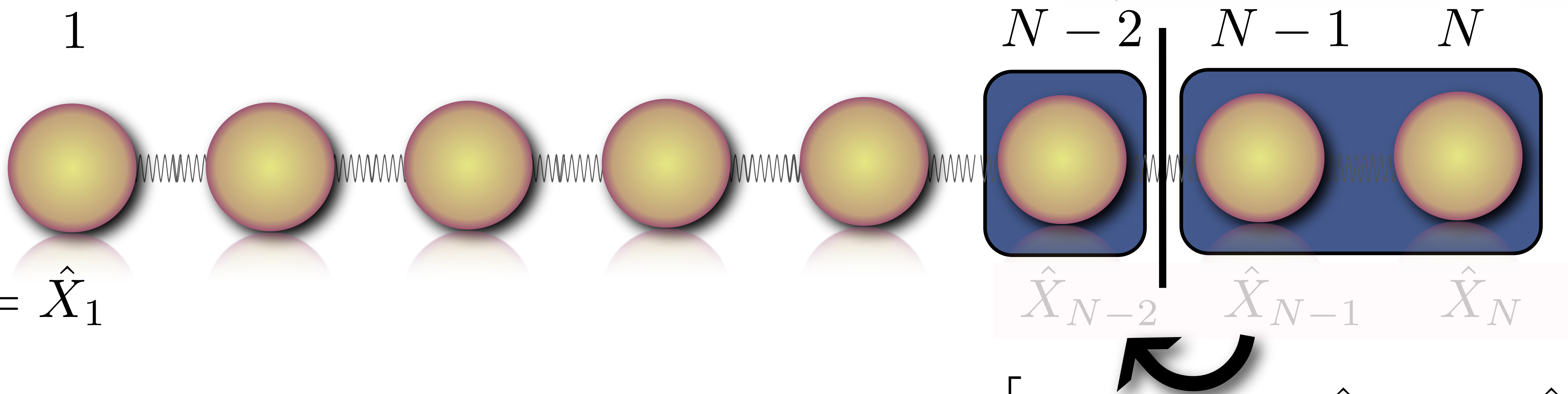

$\hat{O} = \hat{X}_1$

$$\mathrm{tr}_{N-1,N}\left[\hat{\varrho}_{N-2,N-1,N}\,\hat{X}_{N-1}\otimes\hat{X}_N\right]$$

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Efficient State Tomography

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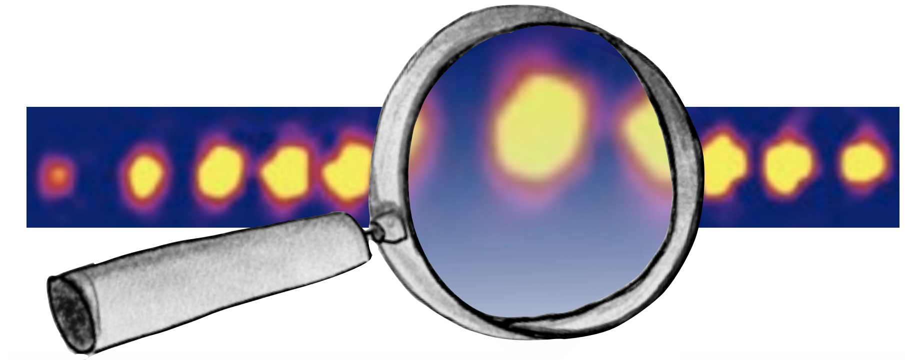

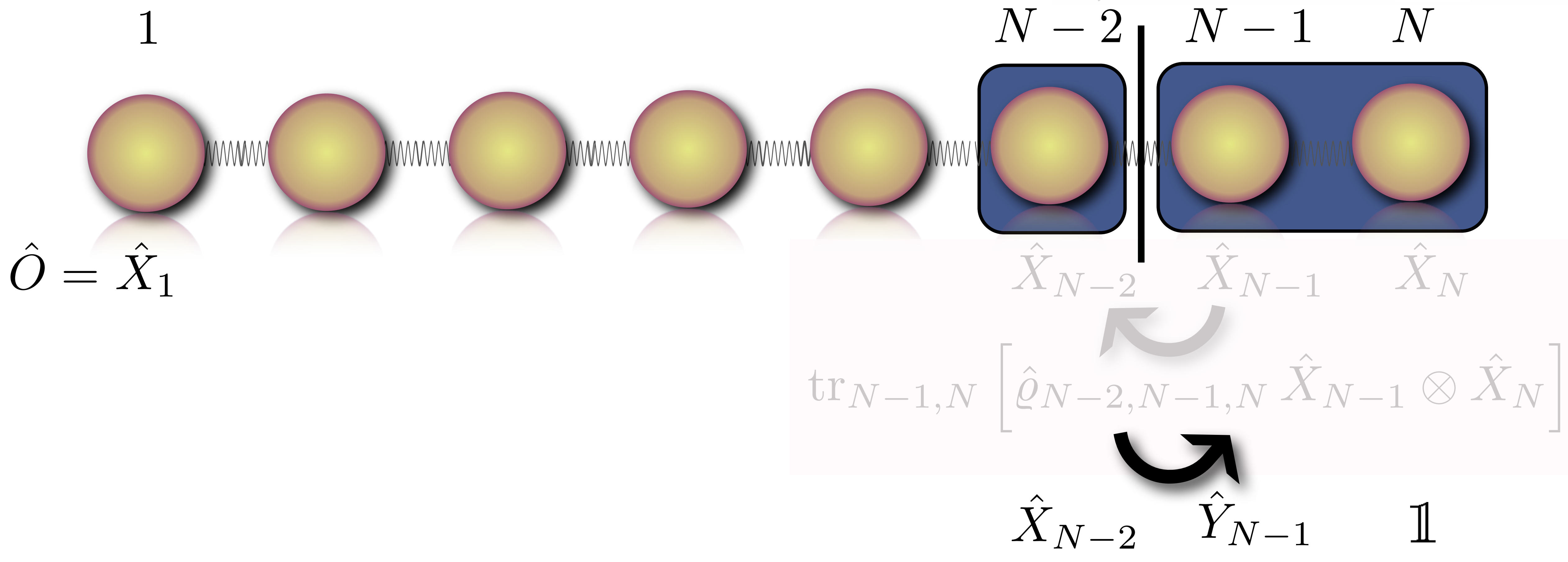

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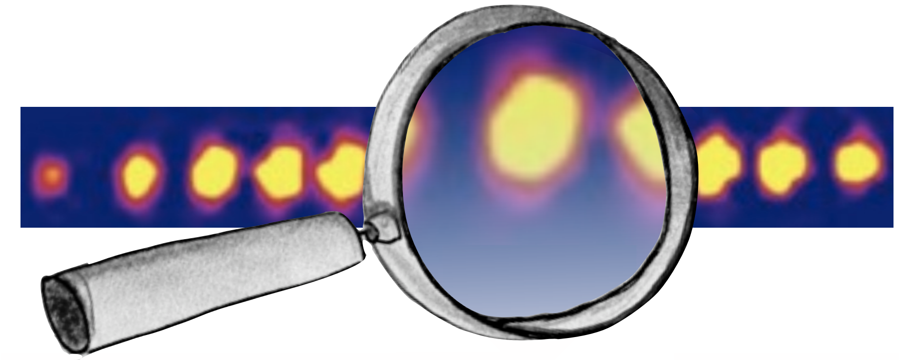

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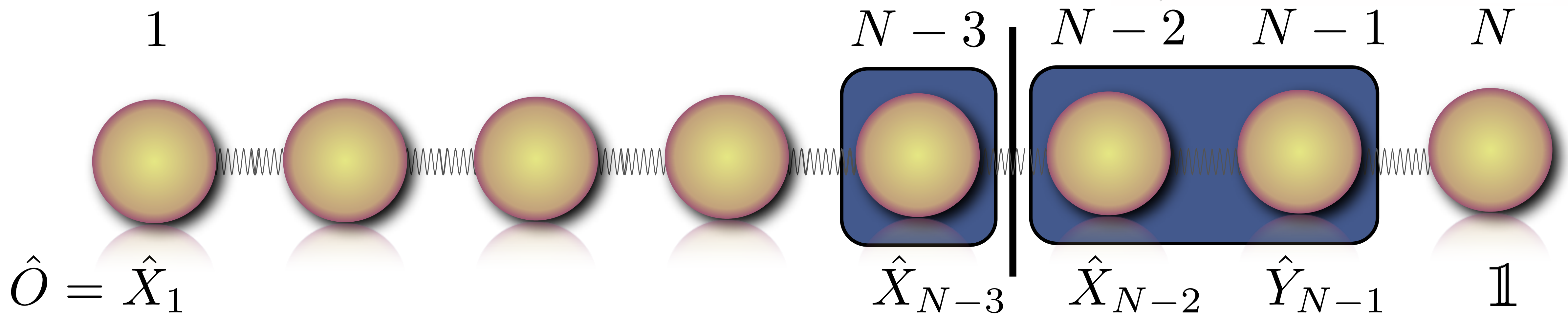

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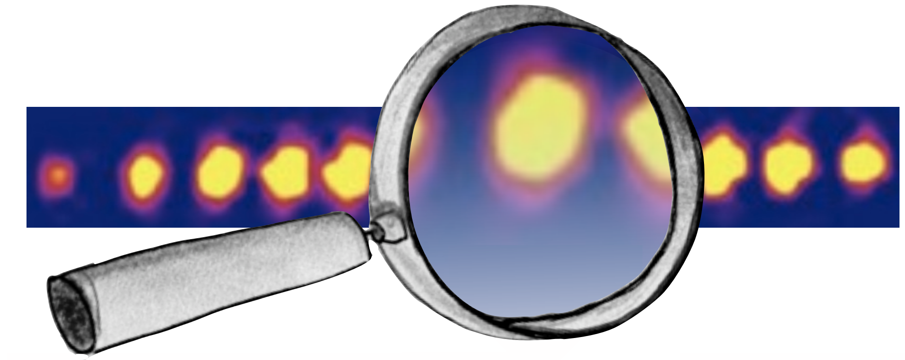
Efficient State Tomography



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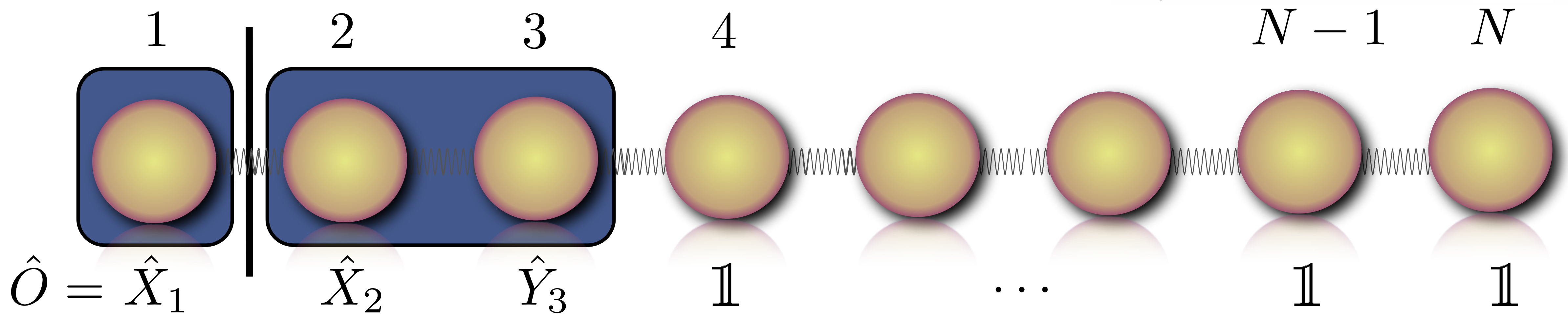


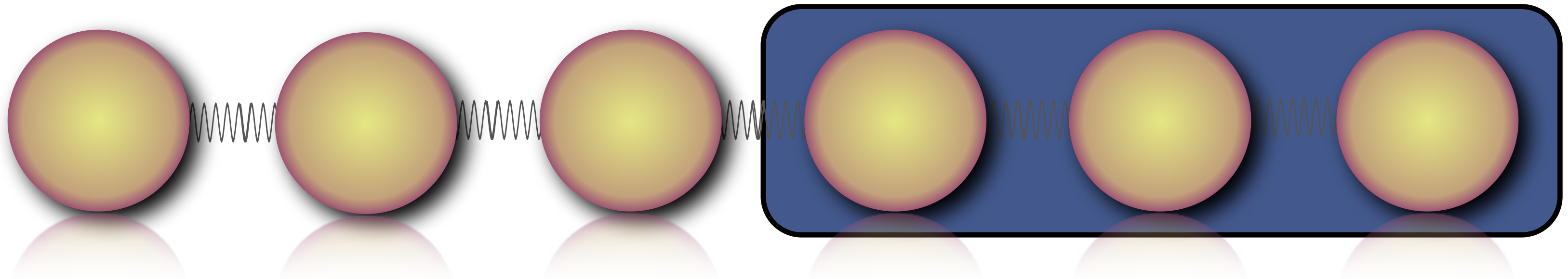
Efficient way to compute expectation values: Choose complete basis.

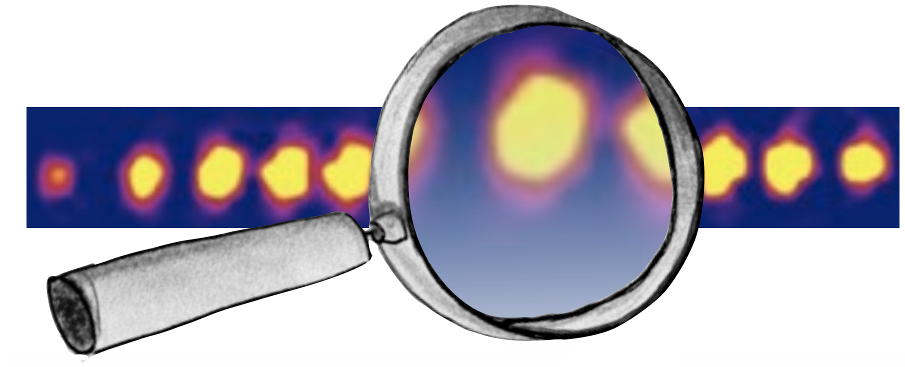
$$\mathrm{tr}\left[\hat{\varrho}\,\hat{P}_1^{(l_1)} \otimes \ldots \otimes \hat{P}_N^{(l_N)}\right] = \mathbb{E}_1(\hat{P}_1^{(l_1)}, \mathbb{E}_2(\ldots \mathbb{E}_{N-1}(\hat{P}_{N-1}^{(l_{N-1})}, \hat{P}_N^{(l_N)})))$$

$$= B_1[l_1] \cdots B_N[l_N]$$

MPO representation: $\hat{\varrho} = \sum_{l_1,\ldots,l_N} B_1[l_1] \cdots B_N[l_N]\, \hat{P}_1^{(l_1)} \otimes \cdots \otimes \hat{P}_N^{(l_N)}$

Efficient State Tomography





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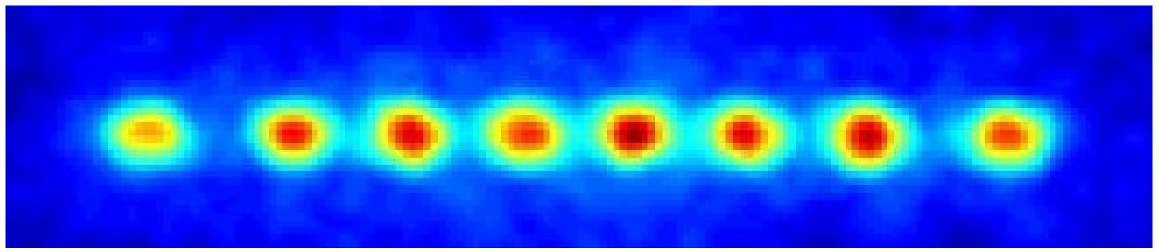
Efficient State Tomography

W state: $|W_{\mathrm{N}=8}(\phi)\rangle = \left[|00\ldots01\rangle + e^{i\phi_1}|00\ldots10\rangle + \ldots + e^{i\phi_{N-1}}|10\ldots00\rangle\right]/\sqrt{N}$

String of $N = 8$ trapped ions:



H. Häffner *et al.*, Nature **438**, 643 (2005).

Full tomography via maximum likelihood: 3^N settings.

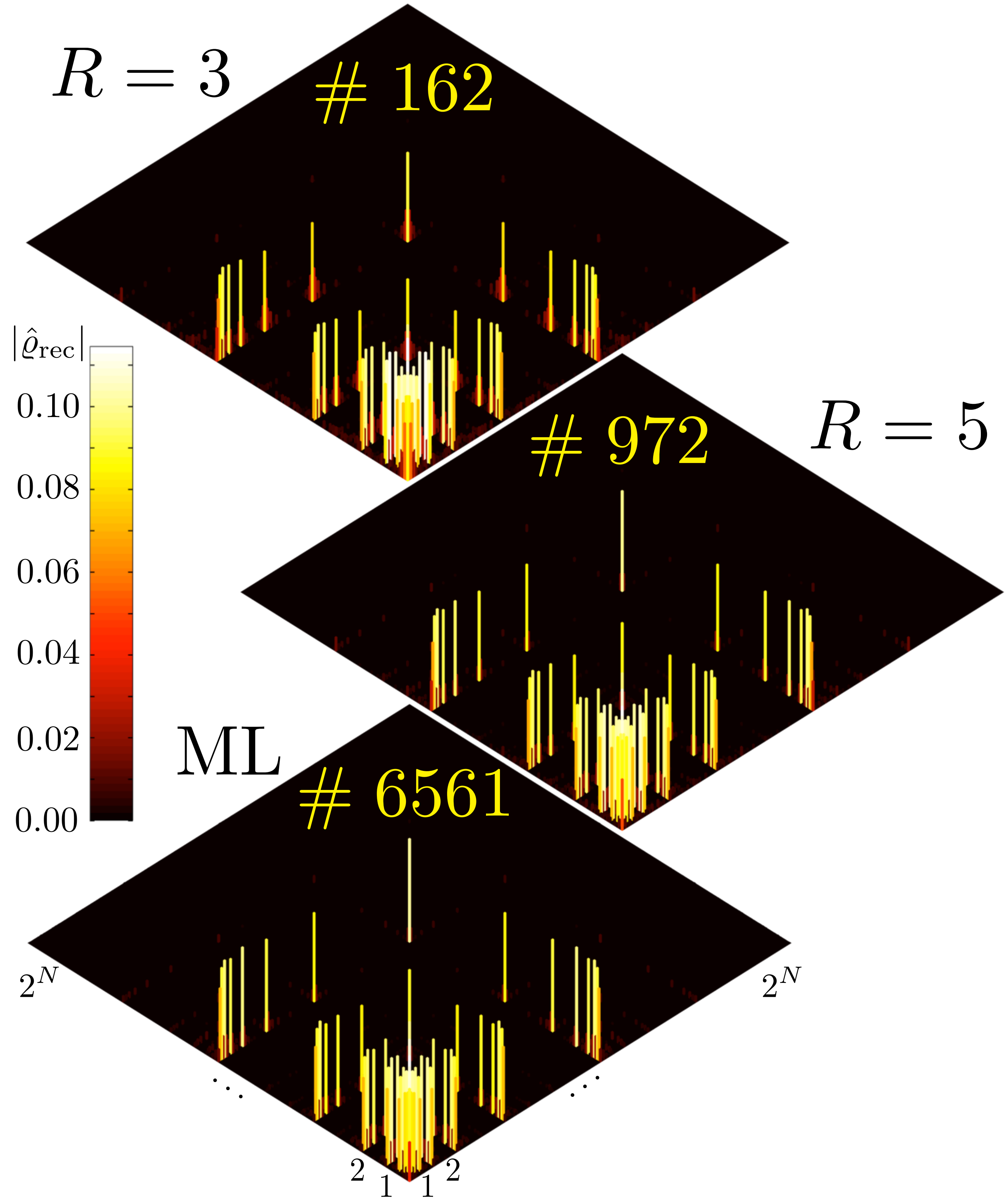
Efficient tomography via direct MPO reconstruction:

$3^R(N - R + 1)$ settings.

Determine maximal fidelity with respect to a W state:

$$f_{\mathrm{opt}} = \underset{\phi}{\mathrm{argmax}}[\langle W_N(\phi)|\hat{\varrho}|W_N(\phi)\rangle]$$

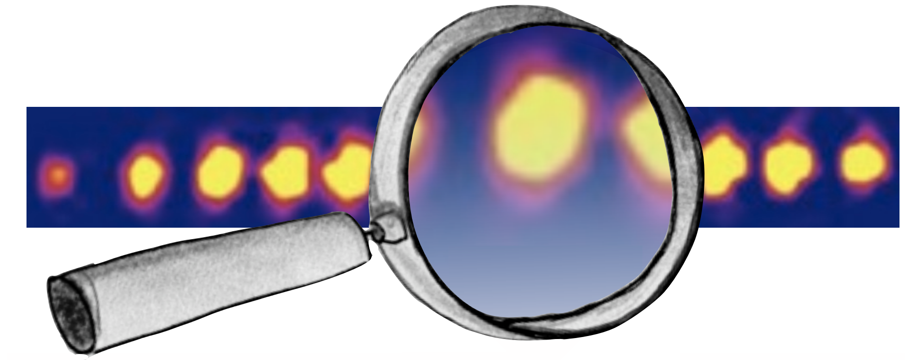
$f_{\mathrm{opt}}^{\mathrm{ML}} = 0.722$ $\quad f_{\mathrm{opt}}^{R=3} = 0.688$ $\quad f_{\mathrm{opt}}^{R=5} = 0.718$



Efficient State Tomography

Ising Hamiltonian:

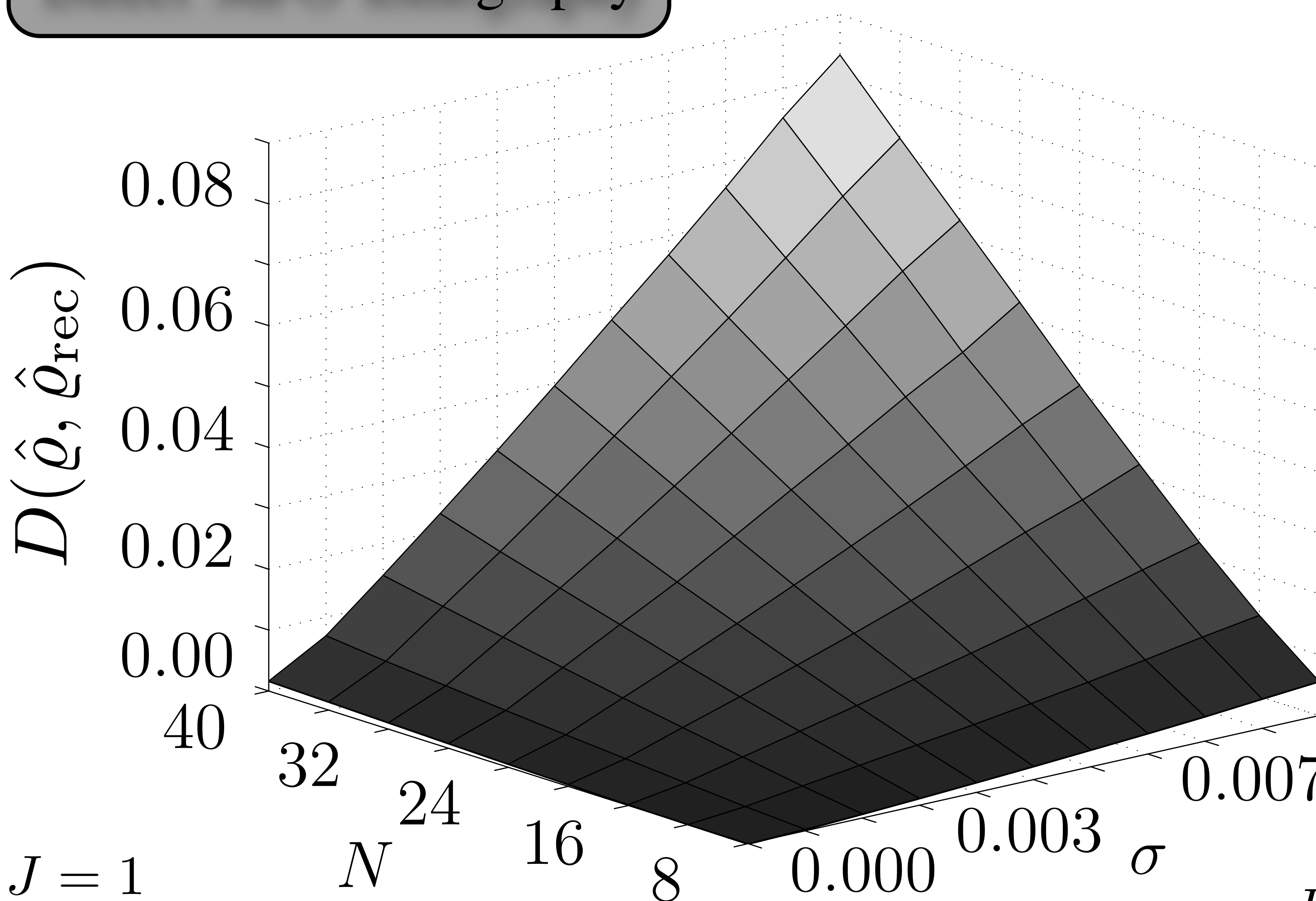
$$\hat{H} = -J \sum_{i=1}^{N-1} \hat{\sigma}_i^{(X)} \otimes \hat{\sigma}_i^{(X)} - B \sum_{i=1}^{N} \hat{\sigma}_i^{(Z)}$$



Consider thermal states:

$$\hat{\varrho} = \mathrm{e}^{-\beta \hat{H}} / Z$$

Direct MPO tomography



Data prone to noise:

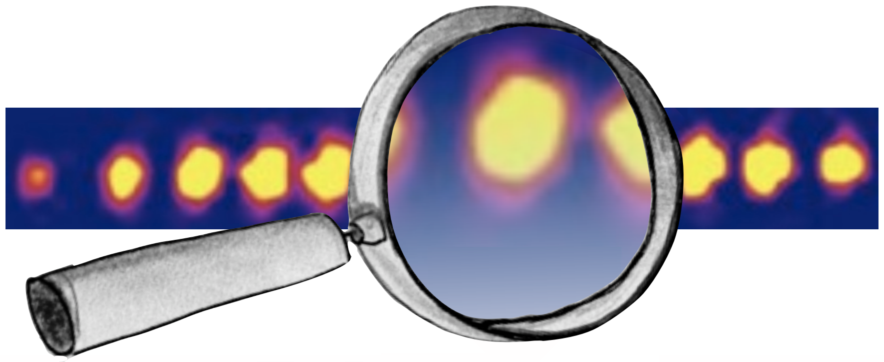
$$p_k = \mathrm{tr}\left[\hat{\varrho}\hat{P}_k\right] + \sigma$$

Gaussian, zero mean.

$J = 1$
$B = 1/2$

$$D(\hat{\varrho}, \hat{\varrho}_{\mathrm{rec}}) = \|\hat{\varrho} - \hat{\varrho}_{\mathrm{rec}}\|_F^2 / \|\hat{\varrho}\|_F^2$$

Efficient State Tomography with Matrix Product Structures



T. Baumgratz

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T. Baumgratz, D. Gross, M. Cramer, and M.B. Plenio, Phys. Rev. Lett. **111**, 020401 (2013).

London, 2nd of June 2014