Entanglement Spectrum and Negativity in Illuminating Impurity Models

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- **Negativity as the Entanglement Measure to Probe the Kondo Regime in the Spin-Chain Kondo Model**
  A. Bayat, P. Sodano, S. Bose, **Phys. Rev. B 81, 064429 (2010)**
- **Entanglement Routers Using Macroscopic Singlets**
  A. Bayat, P. Sodano, S. Bose
  Phys. Rev. Lett. 105, 187204 (2010)
- **An order parameter for impurity systems at quantum criticality**
  A. Bayat, S. Bose, P. Sodano, H. Johannesson, **Nature Communications 5, 3784 (2014)**
- **Entanglement probe of two-impurity Kondo physics in a spin chain**
  A. Bayat, S. Bose, P. Sodano, H. Johannesson, **Phys. Rev. Lett. 109, 066403 (2012)**

Contents of the Talk

Spin Chain Emulation of the Single Impurity Kondo Model

Entanglement (Negativity) to reveal the Kondo Cloud

Kondo Nonequilibrium Dynamics: Entanglement Router

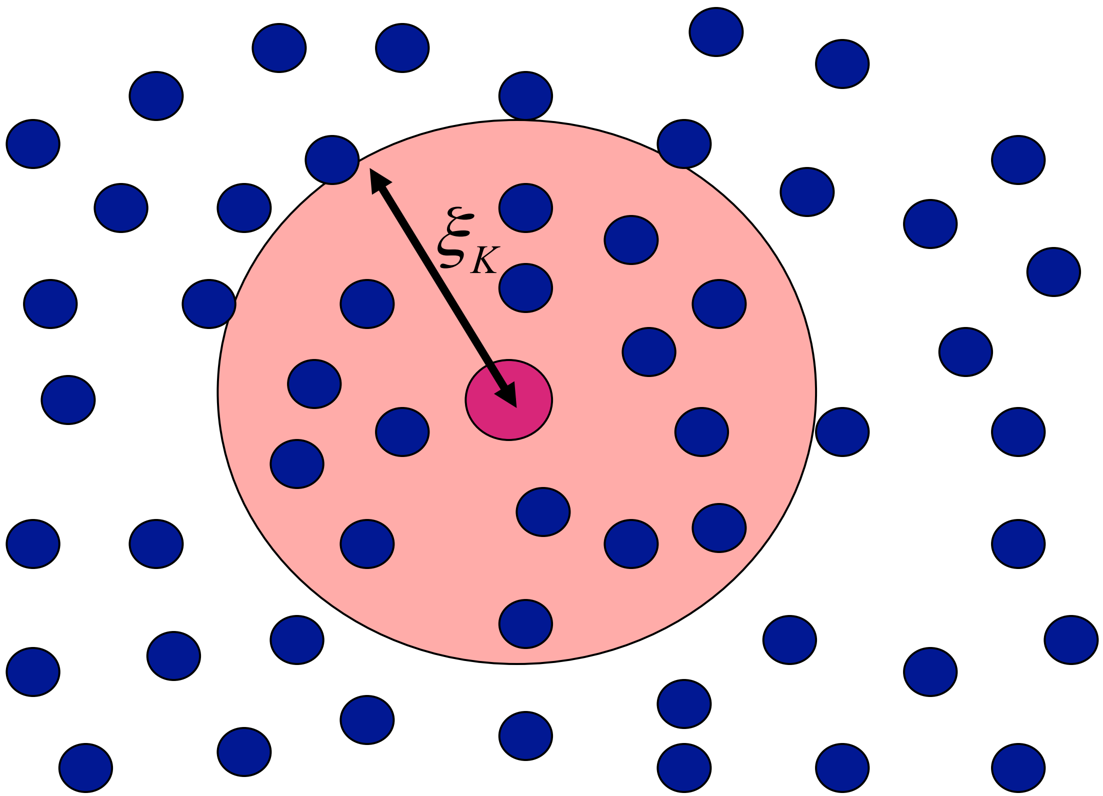
Two Impurity Kondo model (TIKM): Spin Chain Emulation

Real Space Entanglement Structure of TIKM

Entanglement Spectrum of TIKM and Schmidt Gap as an Order Parameter.

Single Impurity Kondo Model & its Spin Chain Version

Kondo Physics



$$\xi_K \Leftrightarrow T_K$$

Despite the gapless nature of the Kondo system, we have a length scale in the model

Realization of Kondo Effect

Semiconductor quantum dots

D. G. Gordon *et al*. Nature 391, 156 (1998).

S.M. Cronenwett, Science 281, 540 (1998).

Carbon nanotubes

J. Nygard, *et al*. Nature 408, 342 (2000).

M. Buitelaar, Phys. Rev. Lett. 88, 156801 (2002).

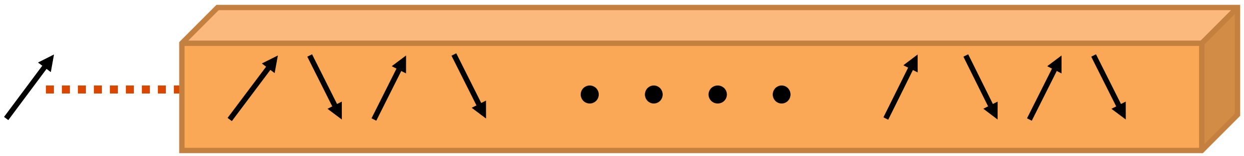
Individual molecules

J. Park, *et al*. Nature 417, 722 (2002).

W. Liang, *et al*, Nature 417, 725–729 (2002).

Evidencing the Kondo Cloud is the "holy grail" of Kondo physics:
L. P. Kouwenhoven and L. I. Glazman, Phys. World 14, 33 (2001).

Kondo Spin Chain



$$H = J'(J_1\sigma_1.\sigma_2 + J_2\sigma_1.\sigma_3) + \sum_{i=2} J_1\sigma_i.\sigma_{i+1} + J_2\sigma_i.\sigma_{i+2}$$

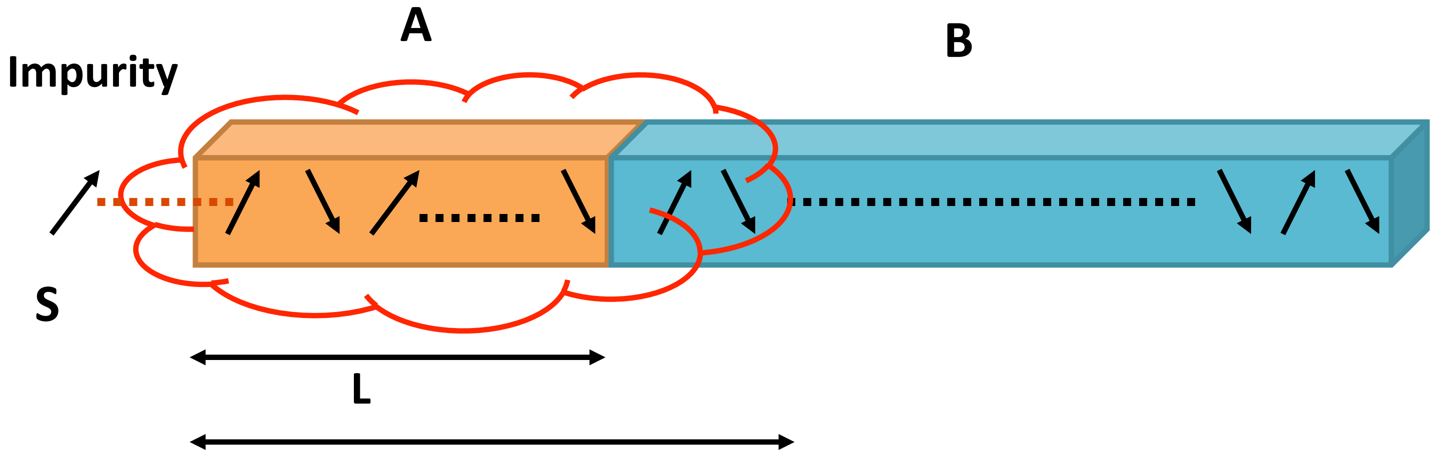
$$\frac{J_2}{J_1} < J_2^c = 0.2412: \quad \text{Kondo (gapless)}$$

$$\frac{J_2}{J_1} > J_2^c: \quad \text{Dimer (gapfull)}$$

E. S. Sorensen *et al.*, J. Stat. Mech., P08003 (2007)

Using only the spin sector of the free electron Kondo Model

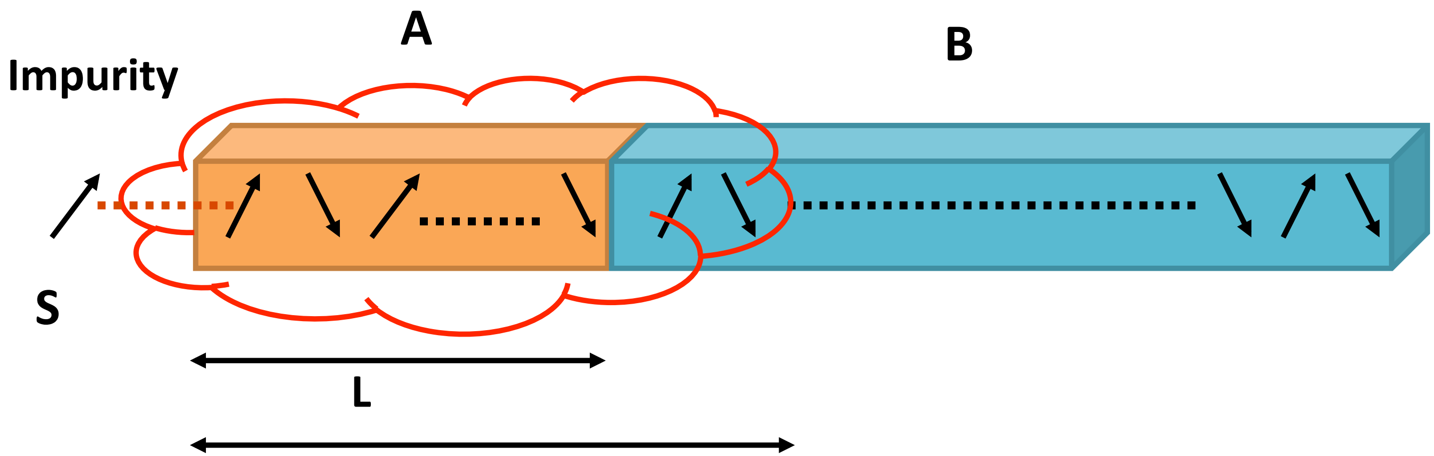
Entanglement as a Witness of the Cloud



$$\xi_K(J') \approx e^{\alpha/\sqrt{J'}}$$

$$\begin{cases} L < \xi_K : \quad E_{SA} < 1 \Rightarrow E_{SB} > 0 \\ L = \xi_K : \quad E_{SA} = 1 \Rightarrow E_{SB} = 0 \\ L > \xi_K : \quad E_{SA} = 1 \Rightarrow E_{SB} = 0 \end{cases}$$

How to *quantify* the entanglement between S and B?



$$\xi_K(J') \approx e^{\alpha/\sqrt{J'}}$$

Negativity

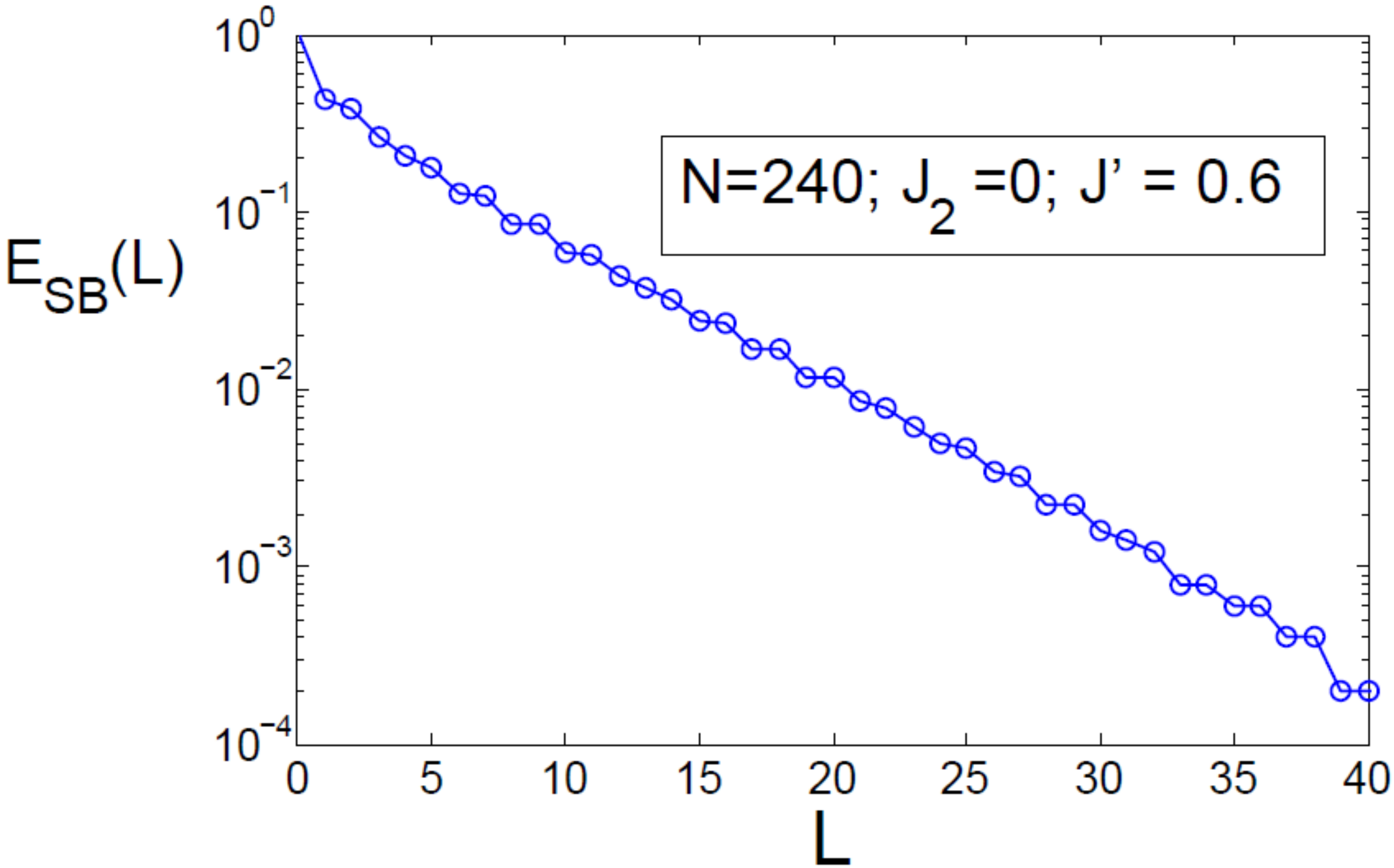
Defined from Peres-Horodecki partial transpose condition by Vidal & Werner in 2002.

$$N(\rho_{SB}) = \sum_i |a_i| - 1$$

$a_i =$ Eigenvalues of $\rho_{SB}^{T_B}$

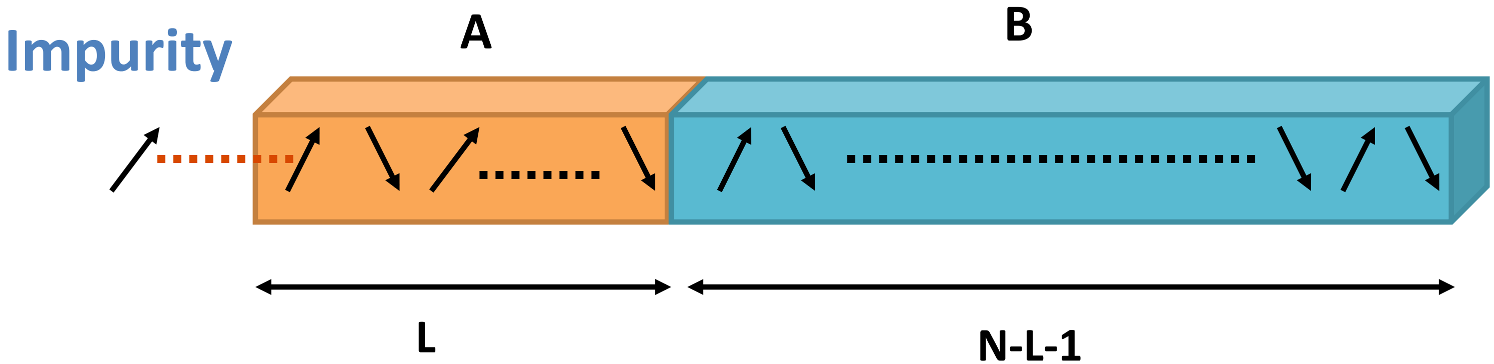
No form of von Neumann entropy suffices!

Entanglement versus Length



Entanglement decays exponentially with length

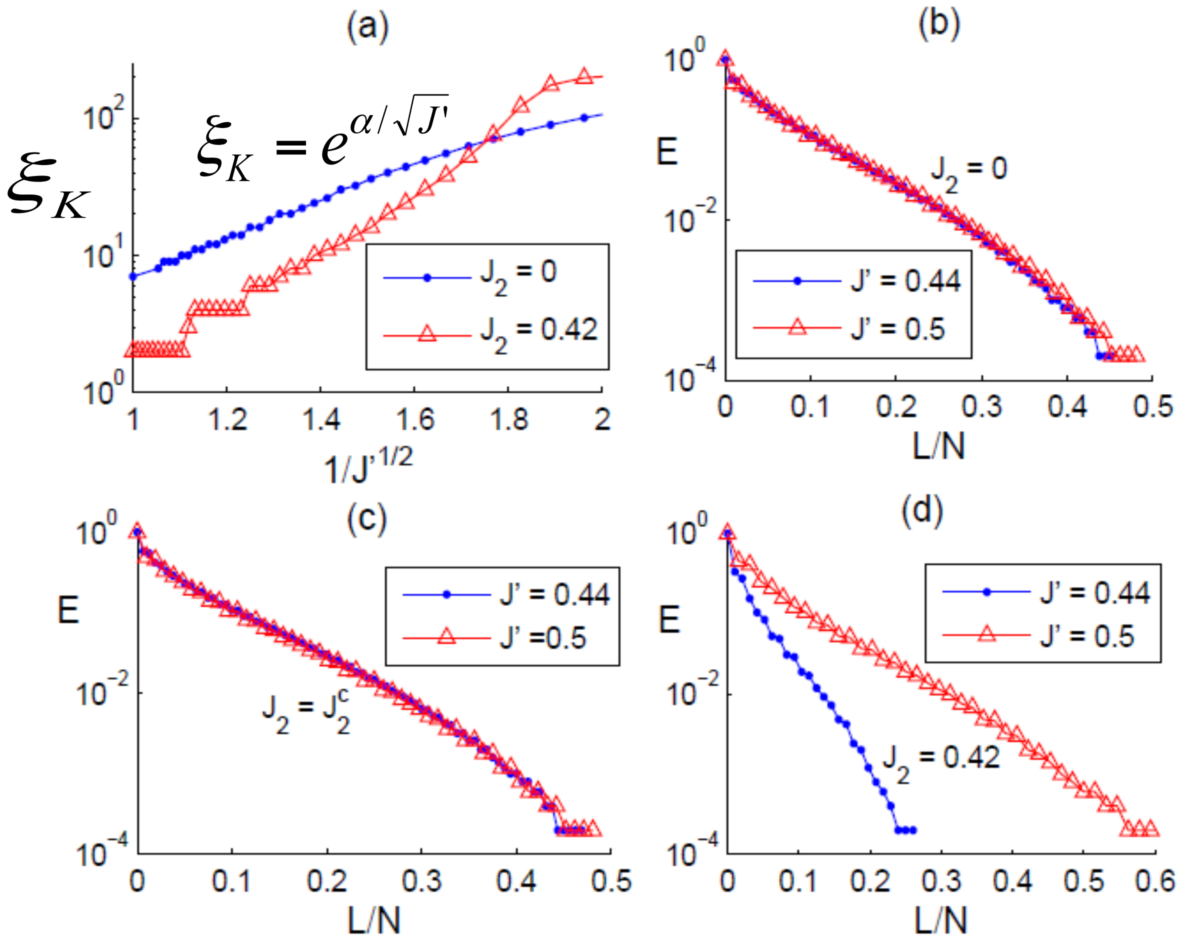
Scaling



Kondo Regime: $$E(L,\xi_K,N)=E(\frac{N}{\xi_K},\frac{L}{N})$$

Dimer Regime: $$E(L,\xi,N)\neq E(\frac{L}{\xi_K},\frac{N}{L})$$

Scaling of the Kondo Cloud



$\frac{N}{\xi_K} = 4$

Kondo Phase: $E(L, \xi_K, N) = E(\frac{N}{\xi_K}, \frac{L}{N})$

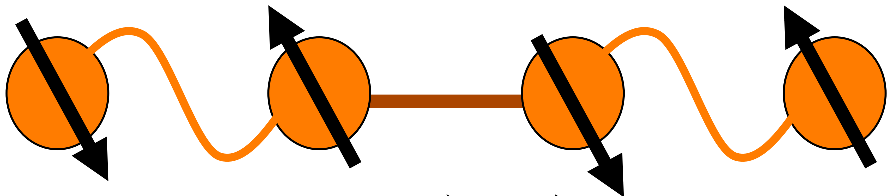
Dimer Phase: $E(L, \xi, N)$

Application: Quantum Router

Converting useless entanglement into useful one through quantum quench followed by nonequilibrium dynamics

Simple Example

$$J'_L\,\vec{\sigma}_1.\vec{\sigma}_2 \qquad J'_R\,\vec{\sigma}_3.\vec{\sigma}_4$$



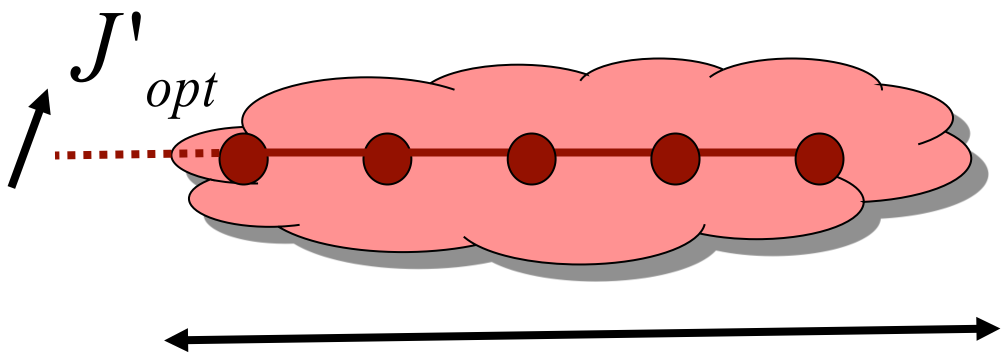
$$J_m\vec{\sigma}_2.\vec{\sigma}_3$$

$$|\psi(0)\rangle = |\psi^-\rangle \otimes |\psi^-\rangle$$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

$$J_m = J'_L + J'_R \quad \Rightarrow \quad E_{14}(t) = \max\{0, \frac{1-3\cos(4J_m t)}{4}\}$$

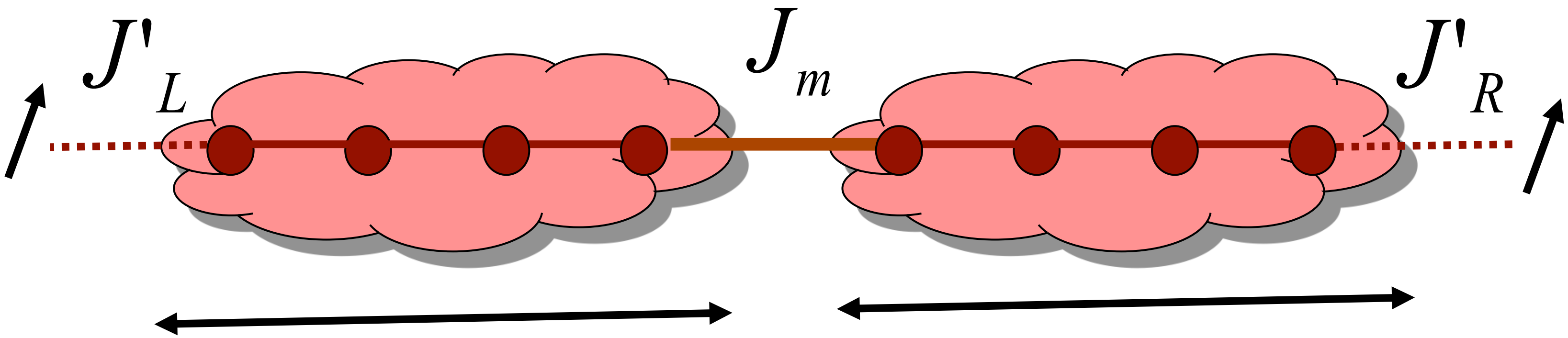
Extended Singlet



$$\xi_K(J'_{opt}) = N - 1$$

With tuning J' we can generate a proper cloud which extends till the end of the chain

Quench Dynamics

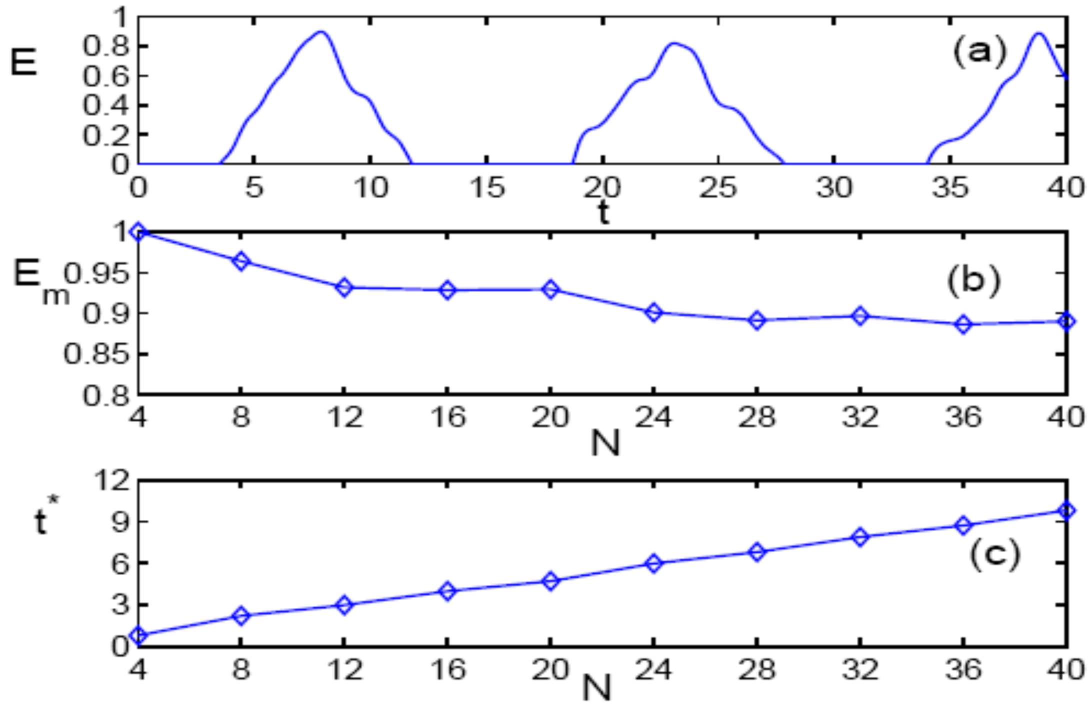


$$\xi_L(J'_L) = N_L - 1 \qquad \xi_R(J'_R) = N_R - 1$$

$$|\psi(0)\rangle = |GS_L\rangle \otimes |GS_R\rangle$$

$$|\psi(t)\rangle = e^{-iH_{LR}t}|\psi(0)\rangle \Rightarrow \rho_{1N}(t) \Rightarrow E_{1N}(t)$$

Attainable Entanglement



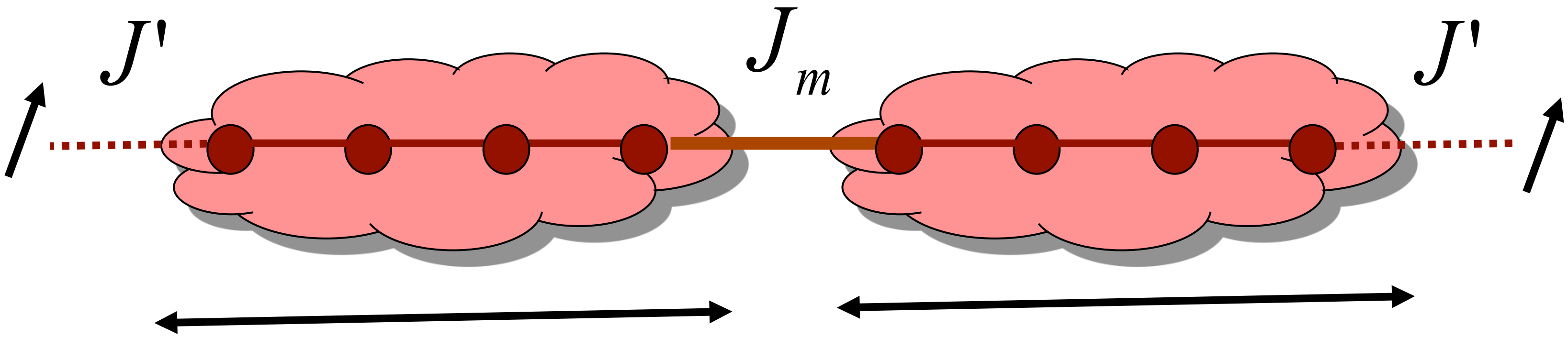
1- Entanglement dynamics is very long lived and oscillatory

2- maximal entanglement attains a constant values for large chains

3- The optimal time which entanglement peaks is linear

Distance Independence

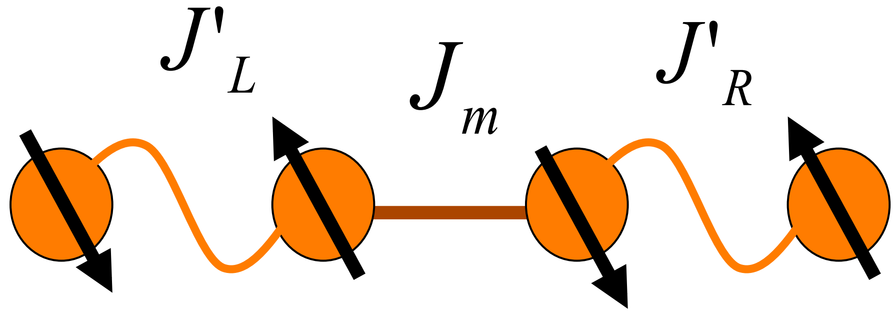
For simplicity take a symmetric composite:



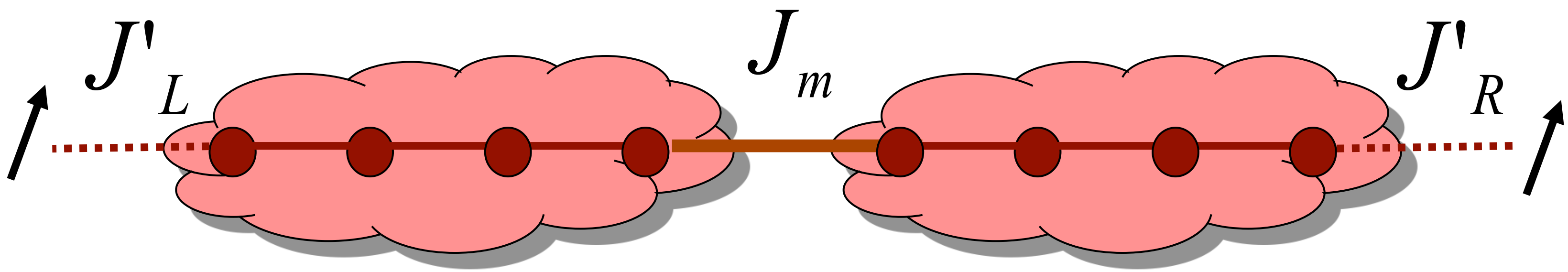
$$\xi_L(J') = (N-2)/2 \qquad \xi_R(J') = (N-2)/2$$

$$E(t,N,J') = E(t,N,\xi) = E(\frac{t}{N},\frac{N}{\xi}) = E(\frac{t}{N},\frac{2N}{N-2})$$

Optimal Quench



$$J_m = J'_L + J'_R$$

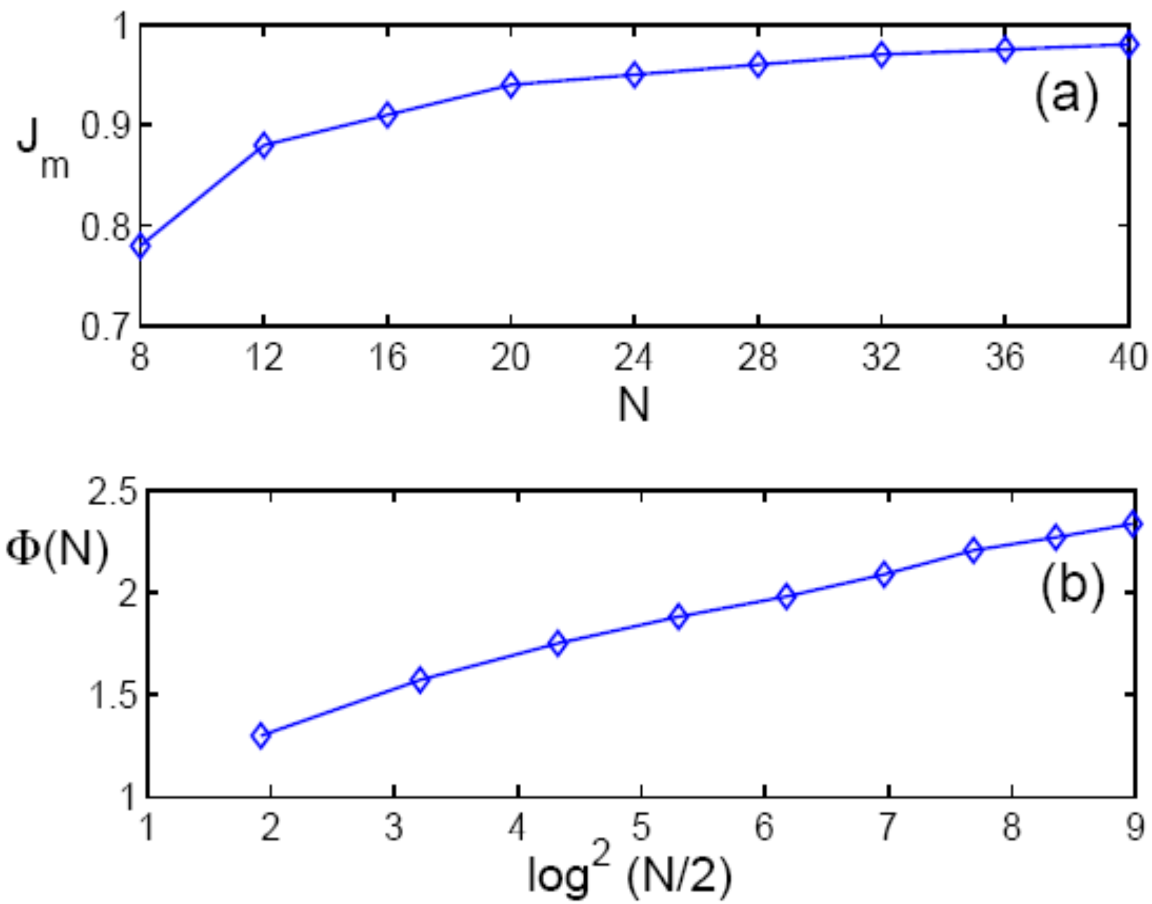


$$J_m = \Phi(N)(J'_L + J'_R)$$

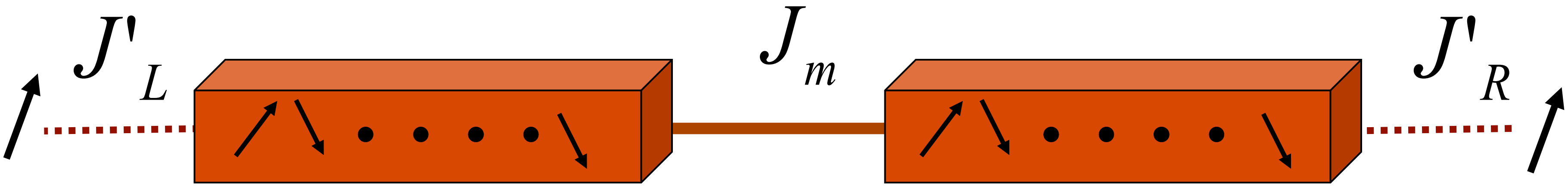
$$\xi_K = e^{\alpha/\sqrt{J'}} \Rightarrow J' \approx \frac{1}{\log^2 \xi}$$

$$\Rightarrow \quad \Phi(N) \approx Log^2\left(\frac{N}{2}\right)$$

Optimal Parameter



Non-Kondo Singlets (Dimer Regime)

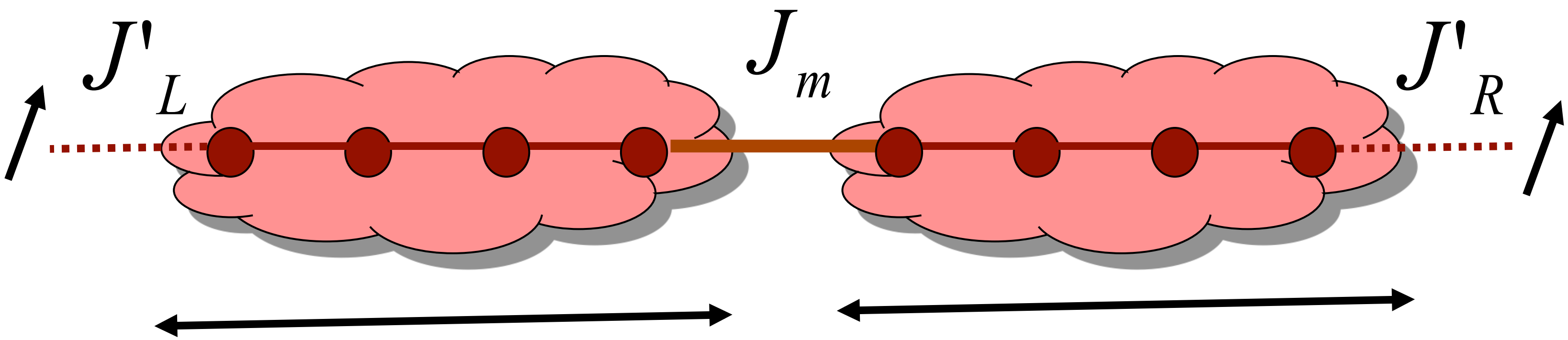


Clouds are absent

N	8	12	16	20	24	28	32	36	40
E_m(K)	0.964	0.932	0.928	0.929	0.901	0.891	0.897	0.886	0.891
E_m(D)	0.957	0.903	0.841	0.783	0.696	0.581	0.468	0.330	0.160
t^*(K)	2.200	2.980	3.980	4.700	5.980	6.800	7.880	8.720	9.800
t^*(D)	3.780	7.290	10.32	13.41	16.89	20.43	24.51	27.12	35.01

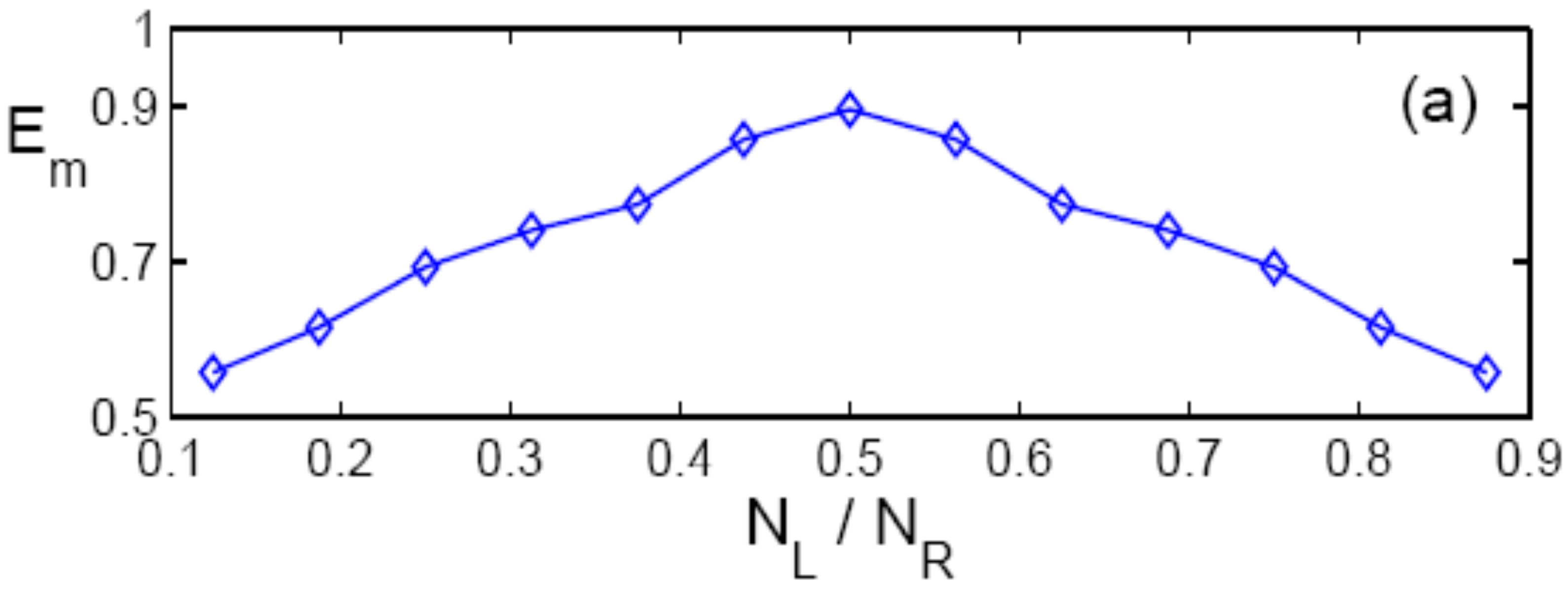
K: Kondo (J2=0) **D: Dimer (J2=0.42)**

Asymmetric Chains



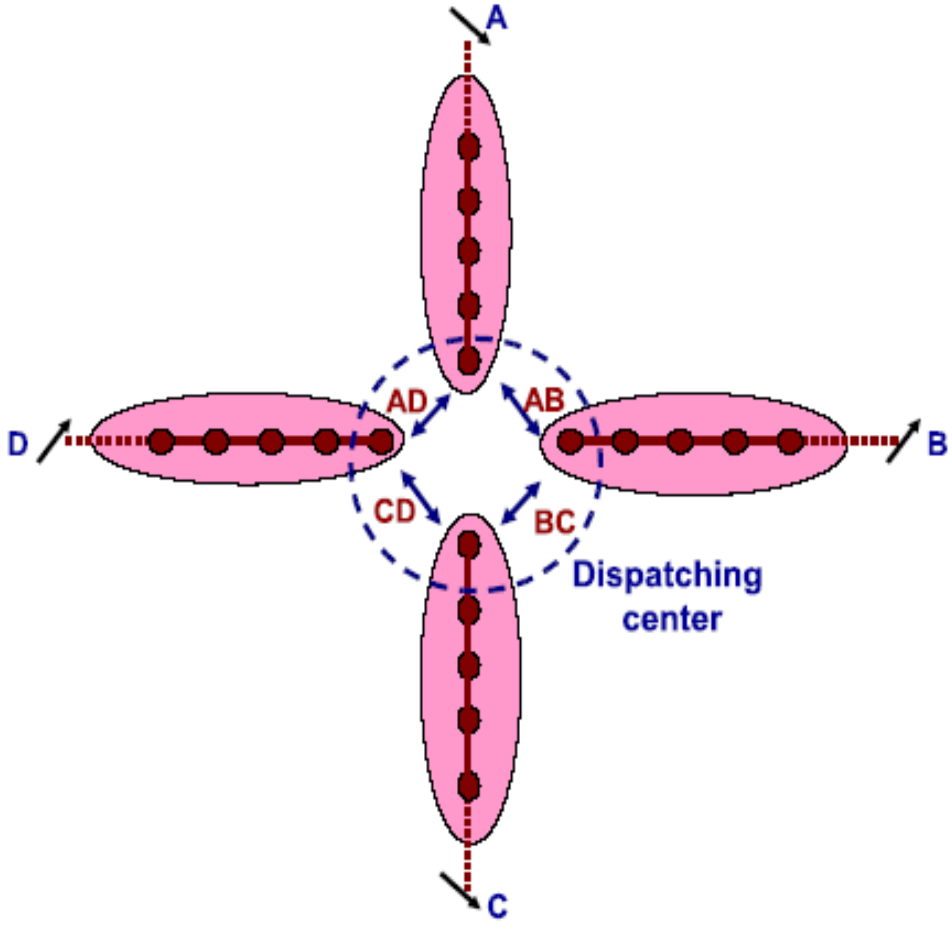
$$\xi_L(J'_L) = N_L - 1 \qquad \xi_R(J'_R) = N_R - 1$$

Entanglement in Asymmetric Chains



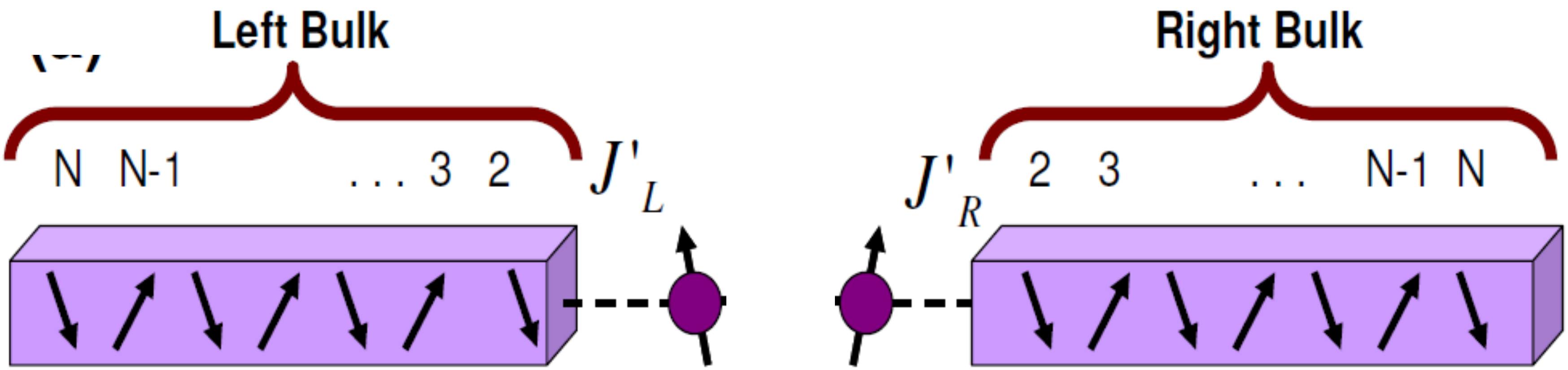
Symmetric geometry gives the best output

Entanglement Router



Two Impurity Kondo Model

Spin Chain Emulation of Two Impurity Kondo Model



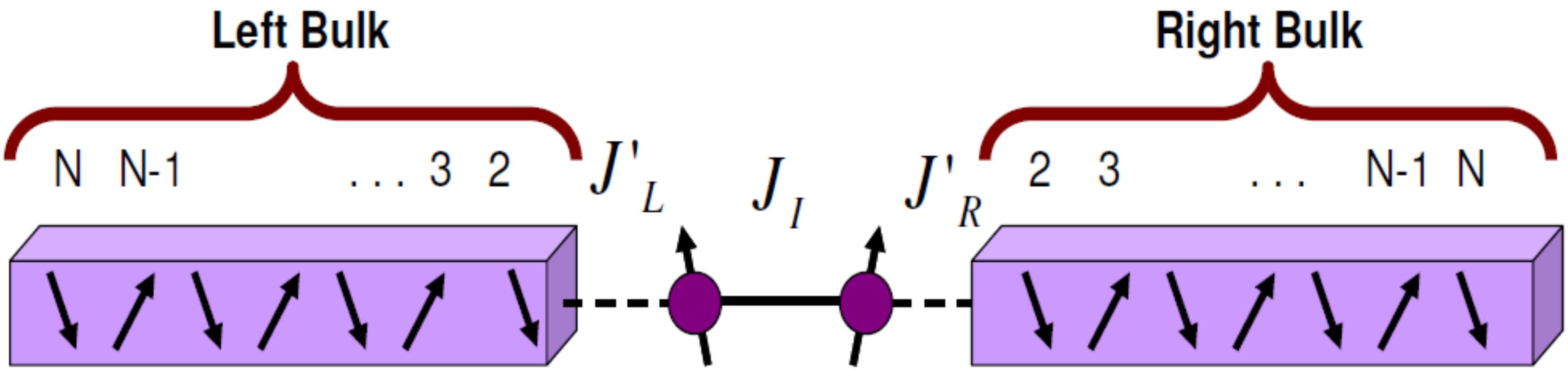
$$H_L = J'(J'_L\,\sigma_1^L.\sigma_2^L + J_2\sigma_1^L.\sigma_3^L) + \sum_{i=2}^{N_L-2} J_1\sigma_i^L.\sigma_{i+1}^L + J_2\sigma_i^L.\sigma_{i+2}^L$$

$$H_R = J'(J'_R\,\sigma_1^R.\sigma_2^R + J_2\sigma_1^R.\sigma_3^R) + \sum_{i=2}^{N_R-2} J_1\sigma_i^R.\sigma_{i+1}^R + J_2\sigma_i^R.\sigma_{i+2}^R$$

$$H_I = J_I\,\sigma_1^L.\sigma_1^R$$

RKKY interaction

Impurities



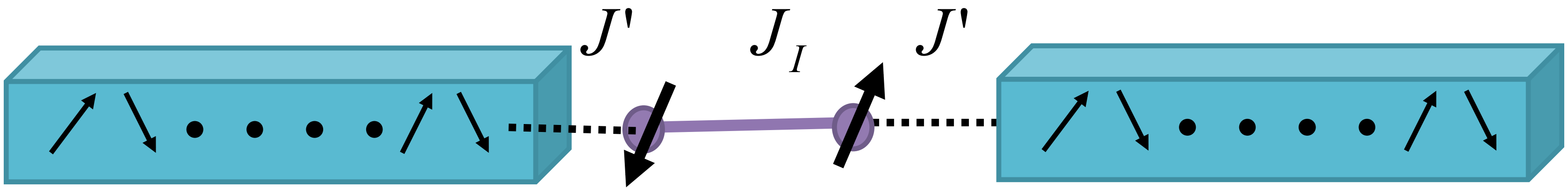
$$|GS\rangle \Rightarrow \rho_{1_L 1_R} = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{3}\sum_{k=0,\pm}|T^k\rangle\langle T^k|$$

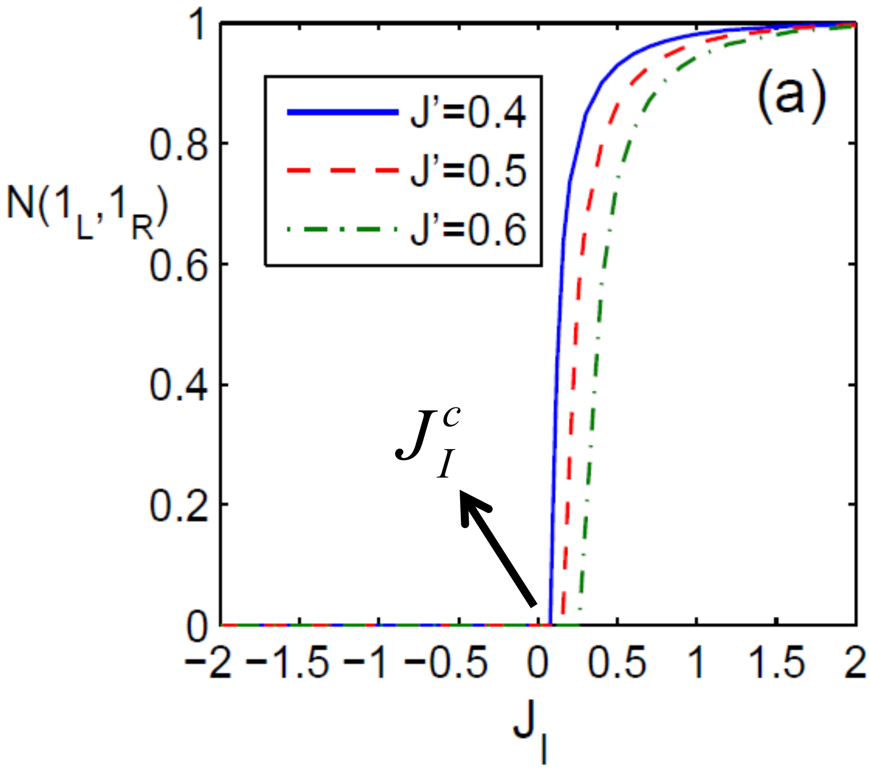
Entanglement

$$p \le \frac{1}{2} \Rightarrow N(\rho_{1_L 1_R}) = 0$$

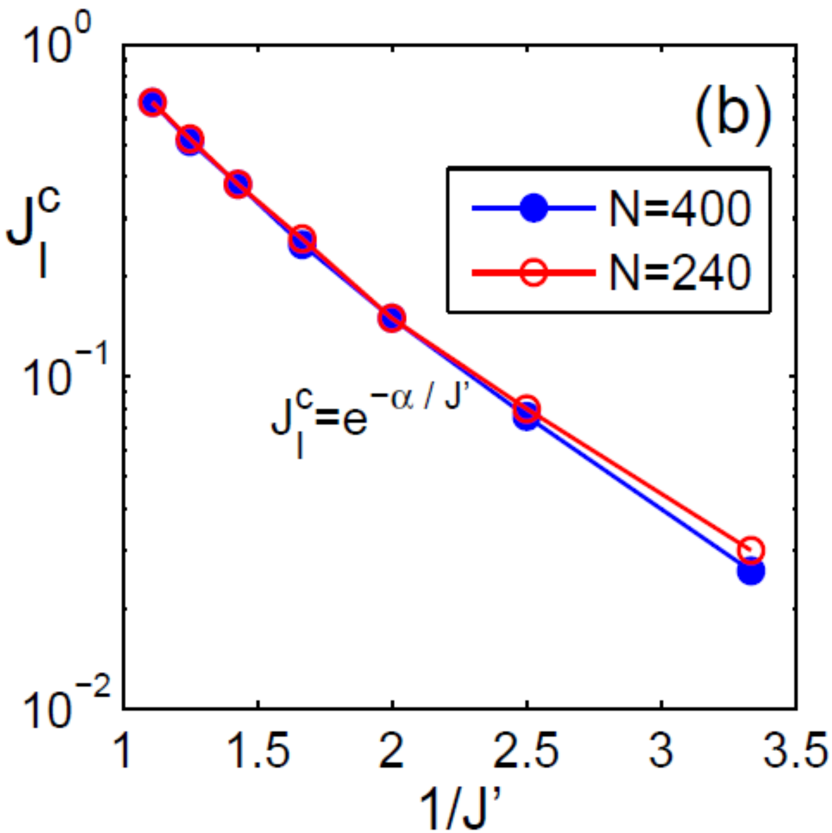
$$p > \frac{1}{2} \Rightarrow N(\rho_{1_L 1_R}) > 0$$

Entanglement of Impurities





Scaling at the Phase Transition



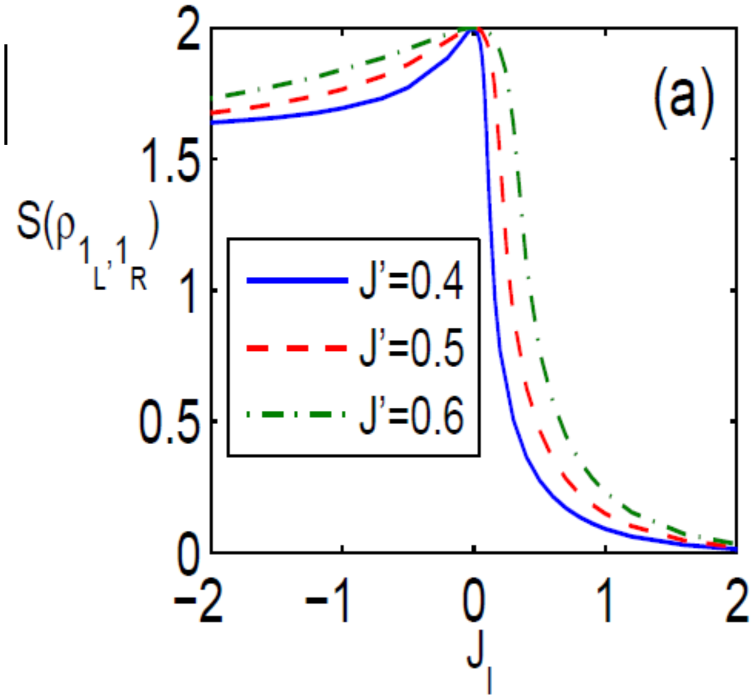
$$J_I^c \propto T_K \propto \frac{1}{\xi_K} \propto e^{-\alpha/J'}$$

The critical RKKY coupling scales just as Kondo temperature does

Entropy of Impurities

$$\rho_{1_L 1_R} = p\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{1-p}{3}\sum_{k=0,\pm}\left|T^k\right\rangle\left\langle T^k\right|$$

$$S(\rho_{1_L,1_R}) = -p\log(p) - (1-p)\log(1-p)$$



$$\begin{cases} J_I << 0: & S(\rho_{1_L,1_R}) = \log(3) \\ J_I = 0: & S(\rho_{1_L,1_R}) = 2 \\ J_I >> 0: & S(\rho_{1_L,1_R}) = 0 \end{cases}$$

⇒

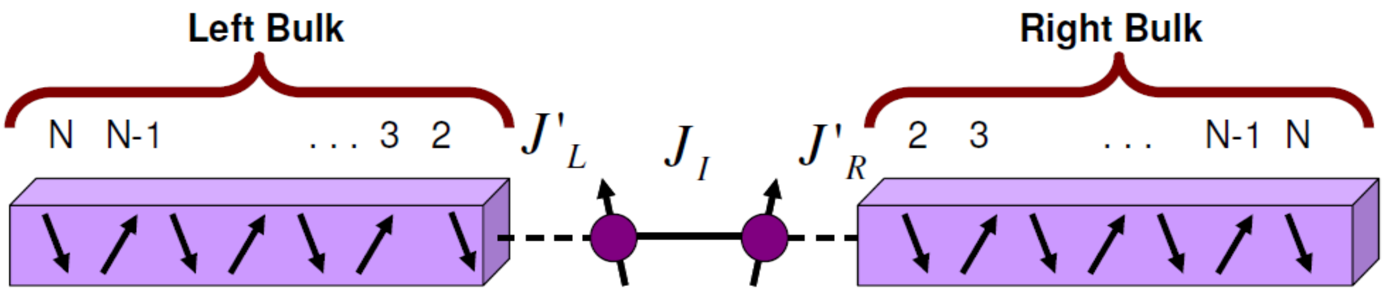
$$\begin{cases} J_I << 0: & p = 0 \\ J_I = 0: & p = 1/4 \\ J_I >> 0: & p = 1 \end{cases}$$

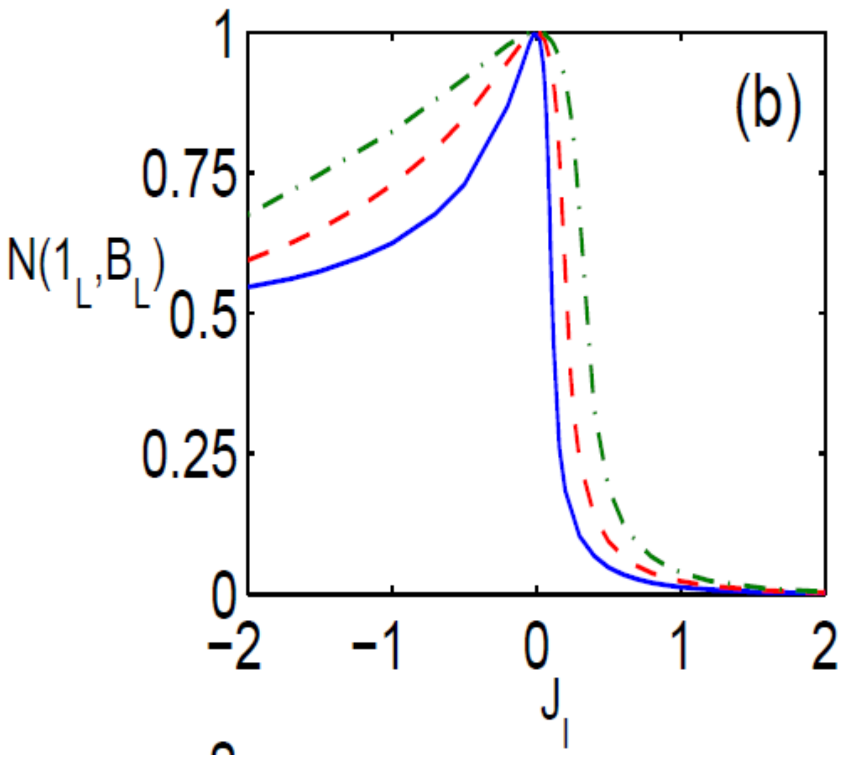
Triplet

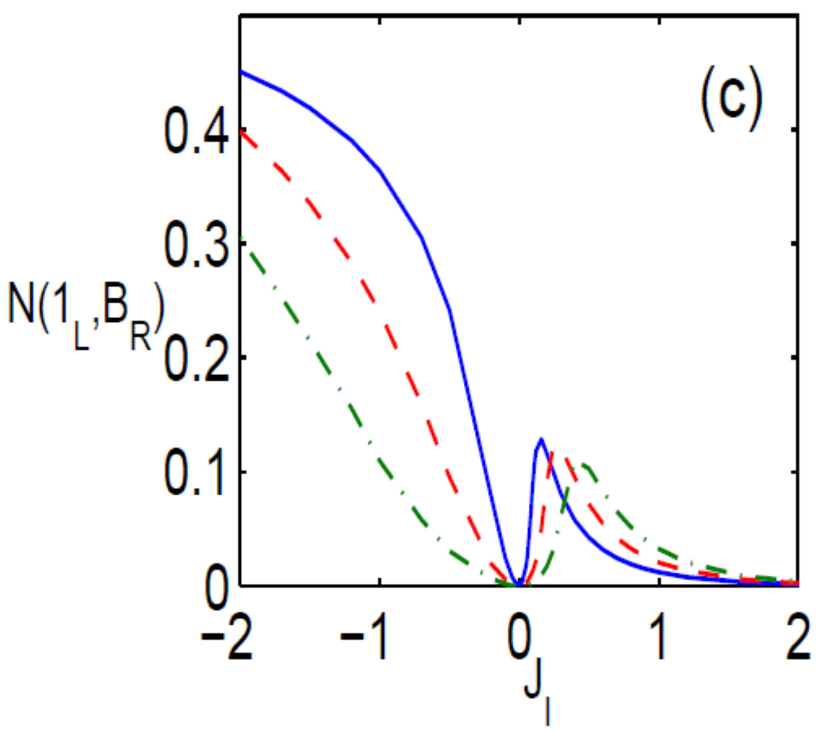
Identity

Singlet

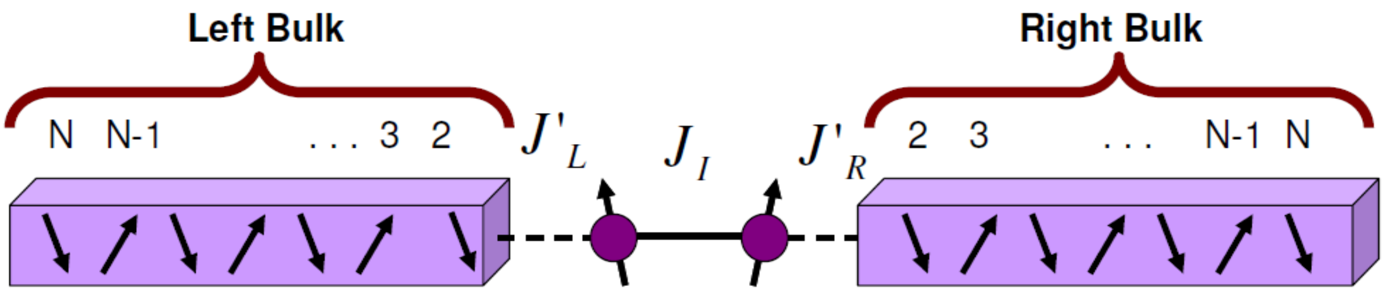
Impurity-Block Entanglement

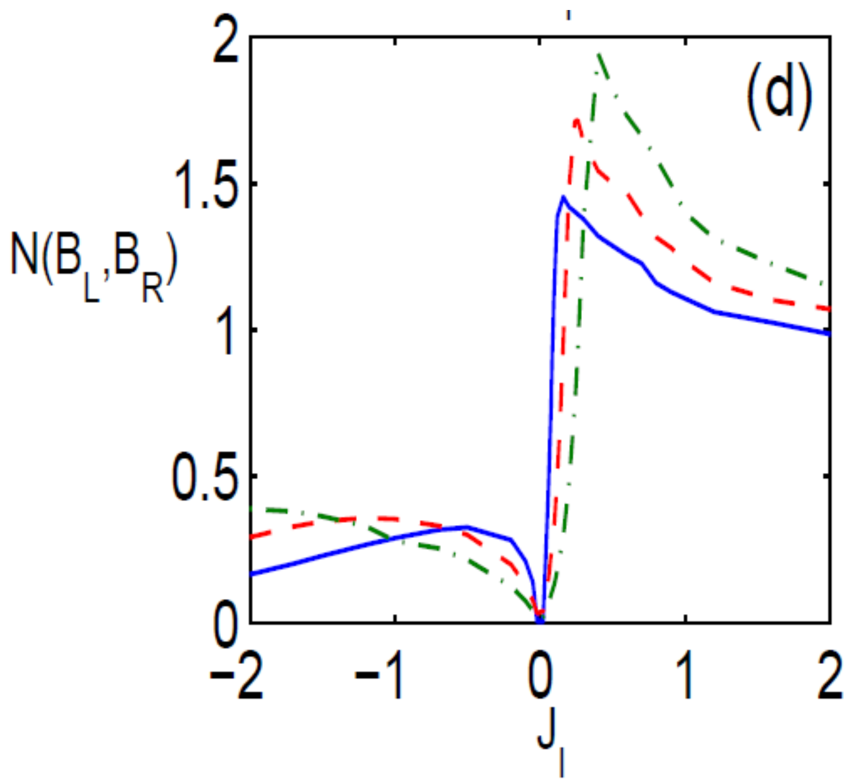




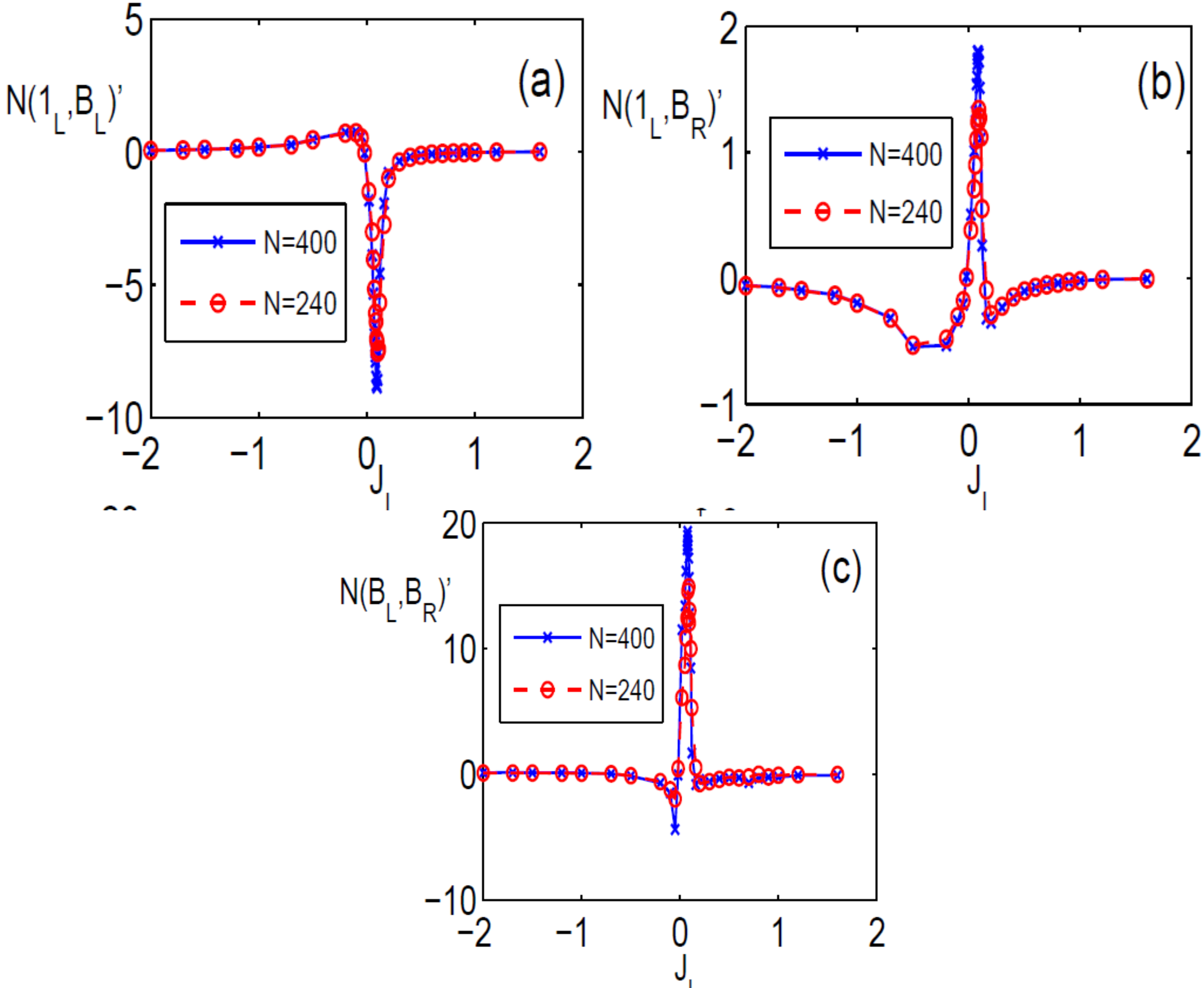


Block-Block Entanglement





2nd Order Phase Transition

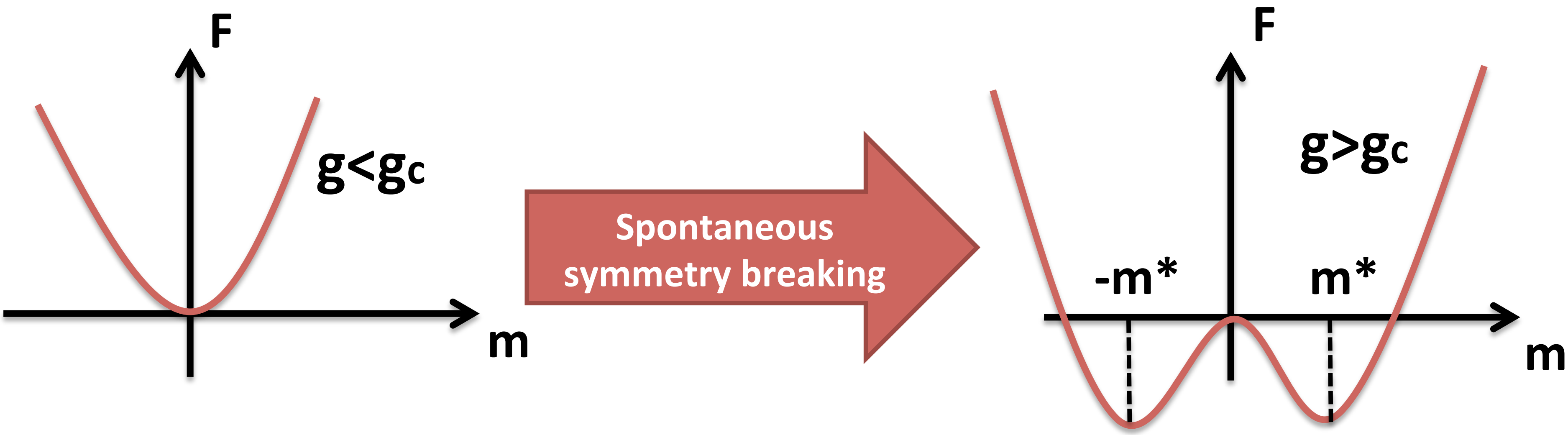


Order Parameter for Two Impurity Kondo Model

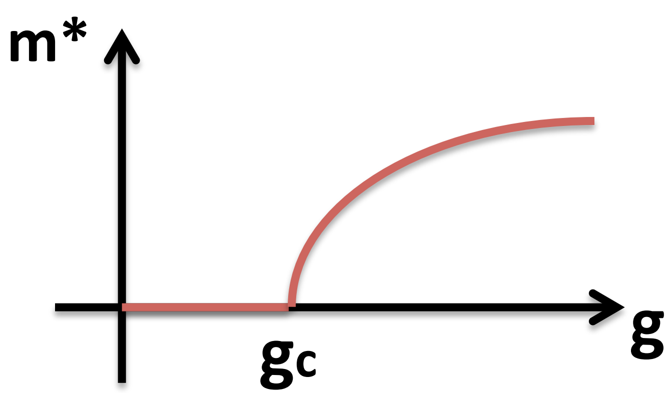
Usual Order Parameter

Phase transition is captured by a local order parameter: $m = \sum_i \sigma_z^i$

Mean field theory for free energy analysis:



Behaviour of the order parameter:



Order Parameter

Order parameter is:

1- Observable
2- Is zero in one phase and non-zero in the other
3- Scales at criticality

Landau-Ginzburg paradigm:

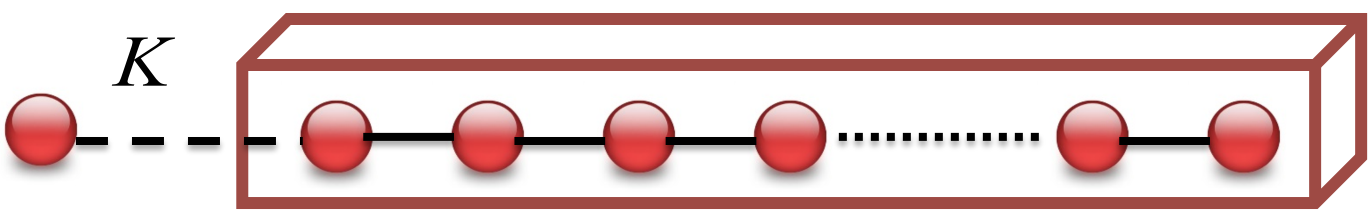
4- Order parameter is local
5- Order parameter is associated with a spontaneous symmetry breaking

Bulk vs. Boundary QPT

Bulk phase transition: a global parameter induces the QPT

$$H_{Ising} = \sum_i \sigma_z^i \sigma_z^{i+1} + B \sum_i \sigma_x^i$$

Boundary phase transition: a local parameter induces the QPT

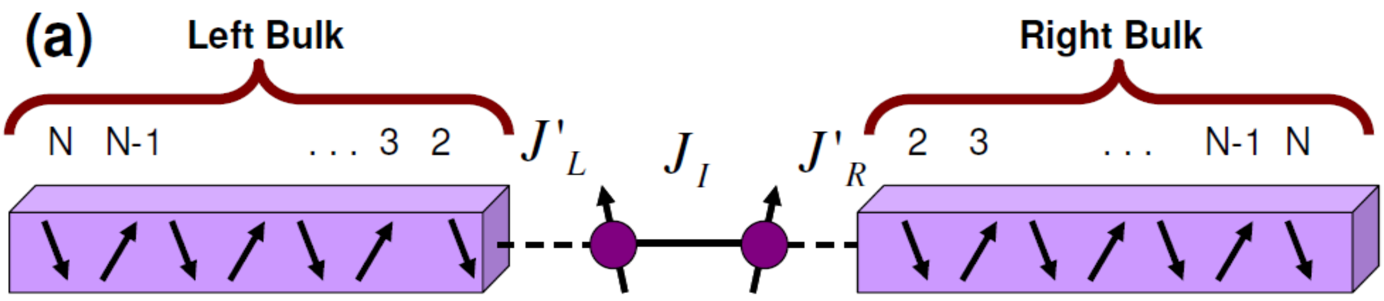


Impurity Phase Transitions

Impurity phase transitions are an example of boundary QPT:

- There is no order parameter (either local or non-local)
- There is no spontaneous symmetry breaking
- RG flows shows unstable fixed points which is an indicator of the QPT

Entanglement Spectrum

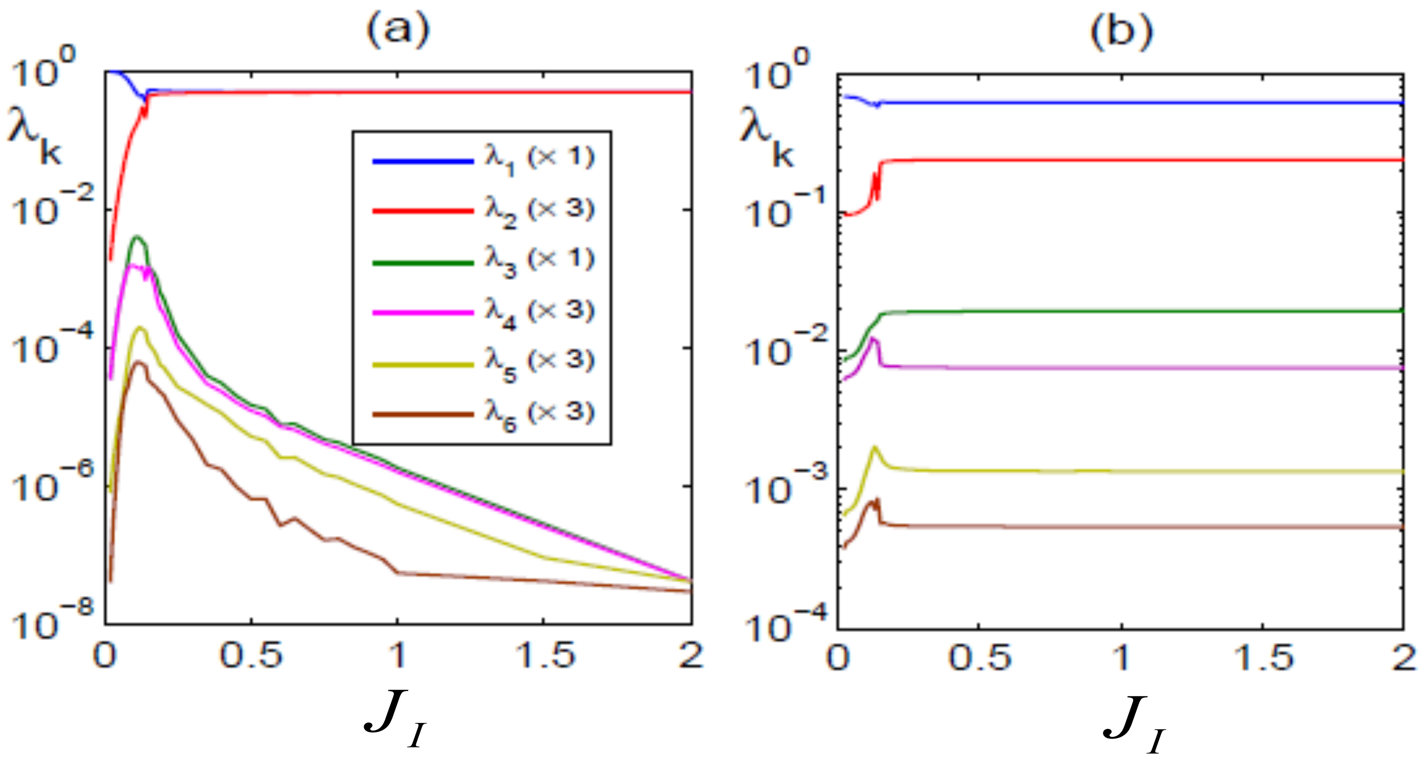


$$|GS\rangle = \sum_k \sqrt{\lambda_k} |A_k\rangle \otimes |B_k\rangle, \quad \lambda_k \geq 0$$

$$\rho_\alpha = \sum_k \lambda_k |\alpha_k\rangle\langle\alpha_k|, \quad \alpha = A, B.$$

Entanglement spectrum: $\lambda_1 \geq \lambda_2 \geq \ldots$

Entanglement Spectrum

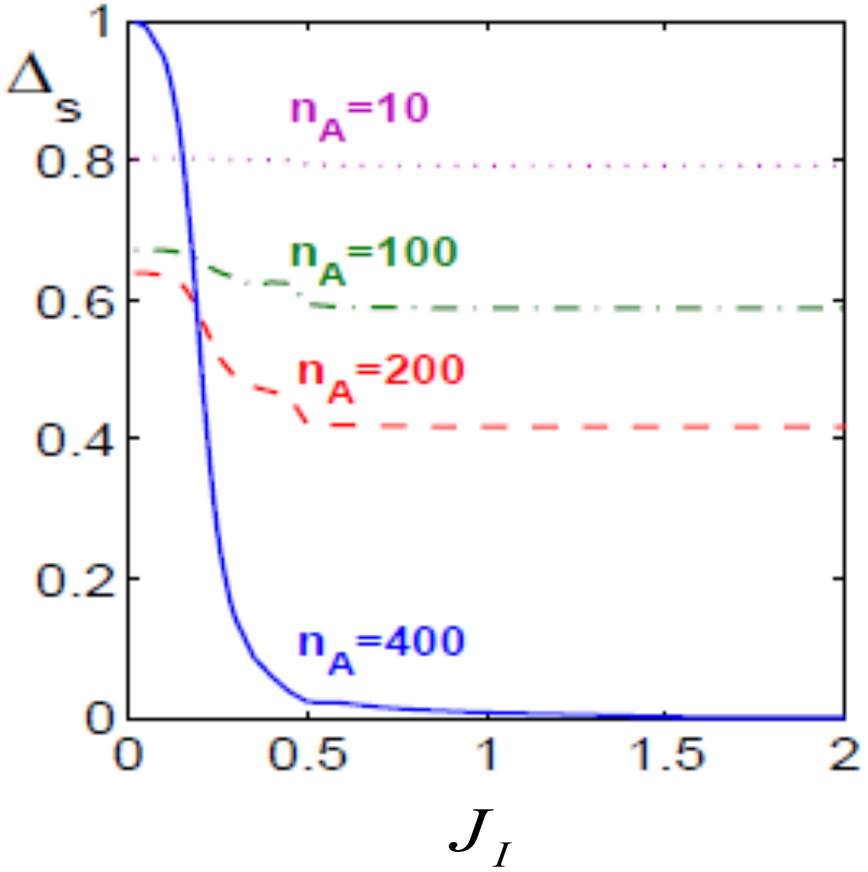


$N_A = N_B = 400$
$J' = 0.4$

$N_A = 600$, $N_B = 200$
$J' = 0.4$

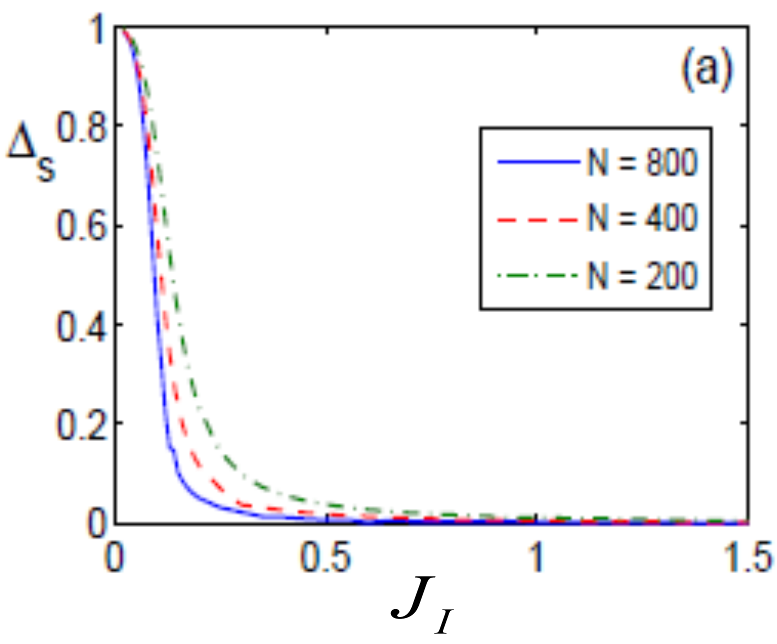
Schmidt Gap

Schmidt gap: $\Delta_S = \lambda_1 - \lambda_2$

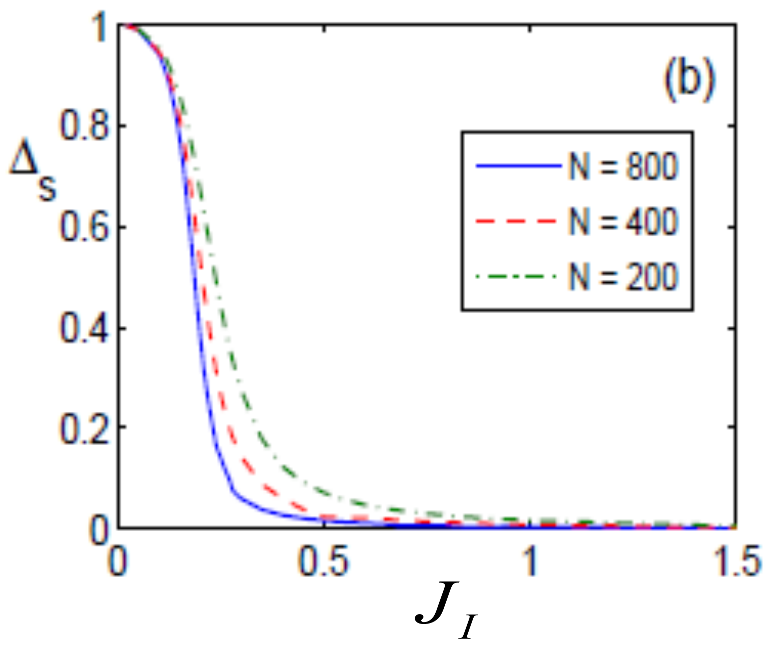


Thermodynamic Behaviour

J’=0.4

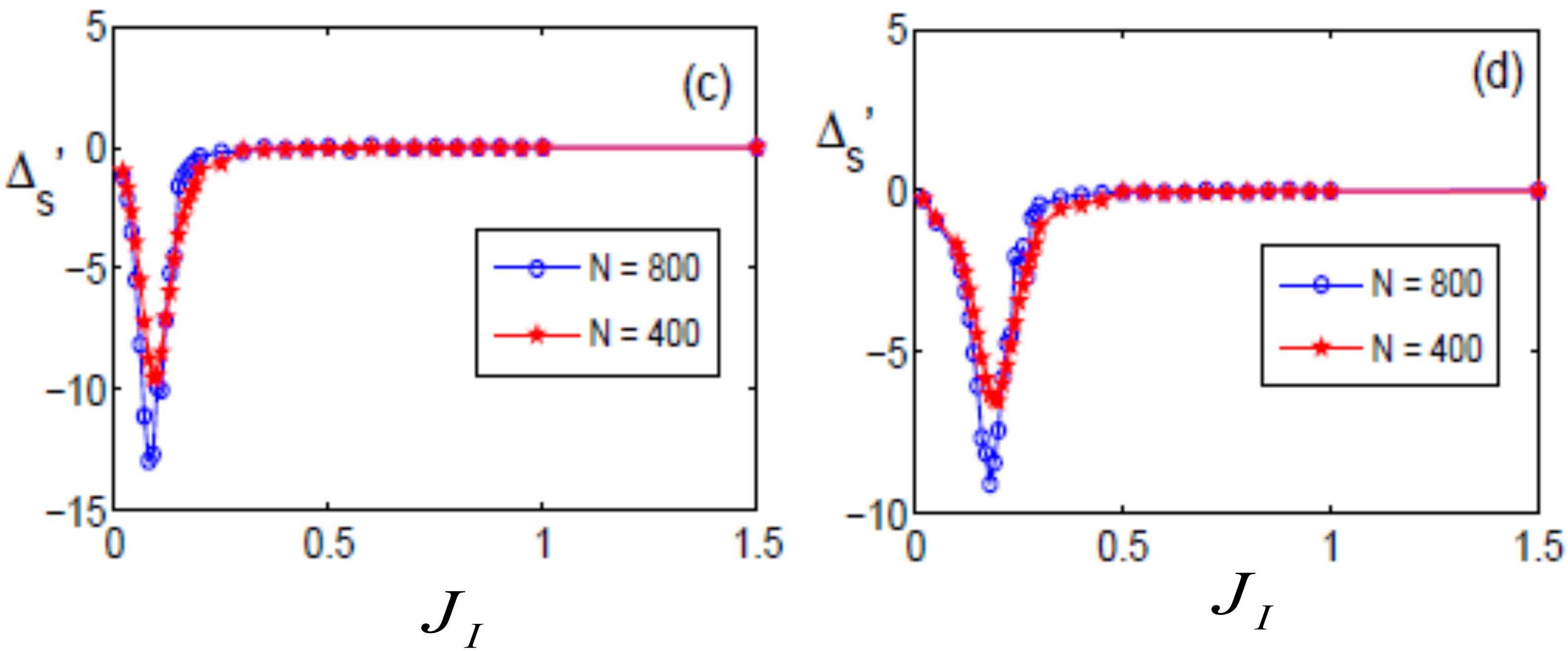


J’=0.5



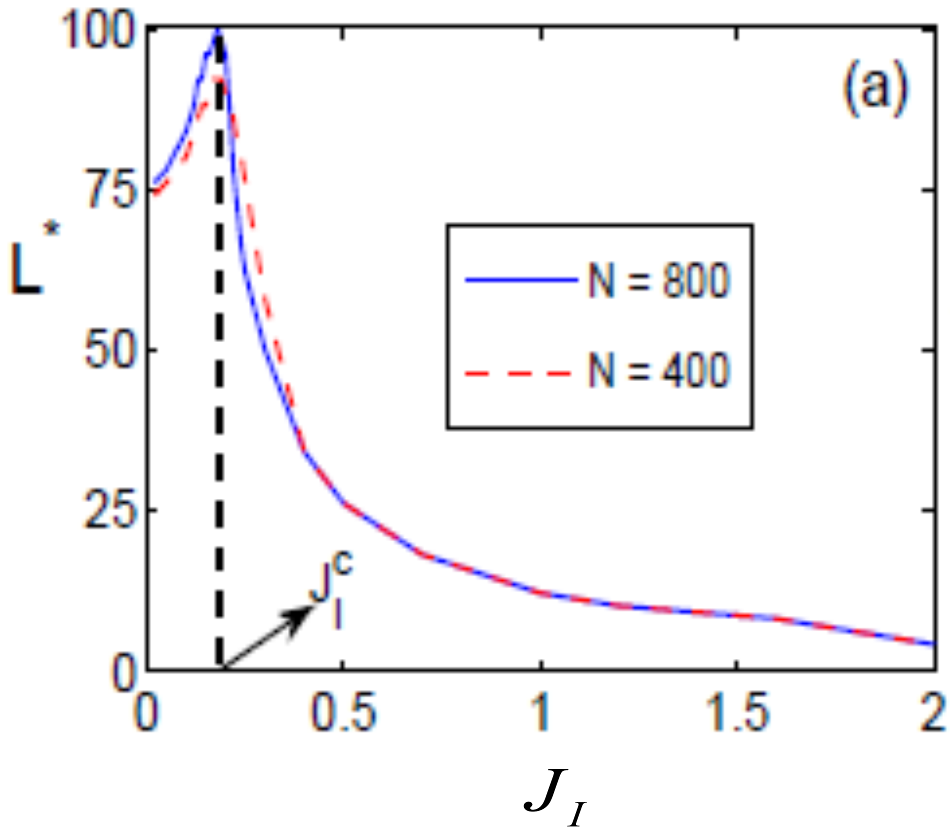
In the thermodynamic limit Schmidt gap takes zero in the RKKY regime

Diverging Derivative



In the thermodynamic limit the first derivative of Schmidt gap diverges

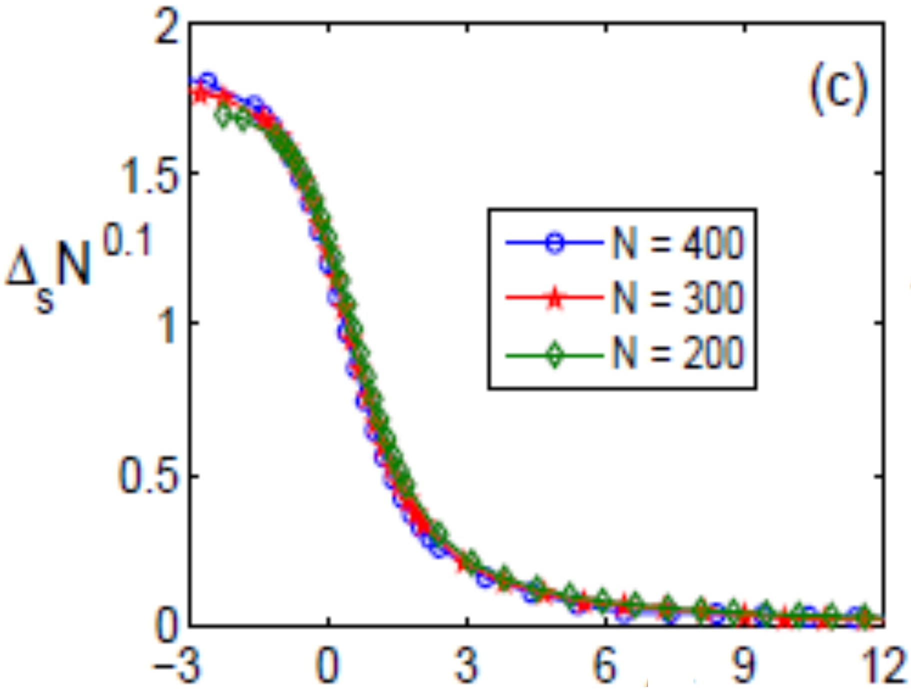
Diverging Kondo Length



Finite Size Scaling

$$\Delta_S = N^{-\beta/\nu} f(|J_I - J_I^c| N^{1/\nu}) \Rightarrow$$

$$\Delta_S N^{\beta/\nu} = f(|J_I - J_I^c| N^{1/\nu})$$



$(J_I - J_I^c) N^{0.5}$

$$\Delta_S = |J_I - J_I^c|^{\beta}$$

$$\xi = |J_I - J_I^c|^{-\nu}$$

$$\beta = 0.2$$

$$\nu = 2$$

Schmidt Gap as an Observable

$$|GS\rangle = \sum_k \sqrt{\lambda_k}|A_k\rangle \otimes |B_k\rangle, \;\; \lambda_k \geq 0$$

$$\mathcal{O} \equiv |A_1\rangle\langle A_1| - |A_2\rangle\langle A_2|$$

$$\langle GS|\mathcal{O}|GS\rangle \;\; = \;\; \lambda_1 - \lambda_2$$

Summary

Impurity systems show exotic quantum phase transition which does not fit in the Landau-Ginzburg paradigm.

Entanglement captures the quantum phase transition in two impurity Kondo model though it is not an order parameter.

Schmidt gap, as an observable, shows scaling with the right exponents at the critical point of the two Impurity Kondo model.

Kodno physics provide distance independent entanglement through a single bond quench.

References

- **An order parameter for impurity systems at quantum criticality**
  A. Bayat, S. Bose, P. Sodano, H. Johannesson
  Nature Communications 5, 3784 (2014)

- **Entanglement probe of two-impurity Kondo physics in a spin chain**
  A. Bayat, S. Bose, P. Sodano, H. Johannesson
  Phys. Rev. Lett. 109, 066403 (2012)

- **Negativity as the Entanglement Measure to Probe the Kondo Regime in the Spin-Chain Kondo Model**
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