Entanglement negativity and quantum field theory

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Joint work with John Cardy and Erik Tonni
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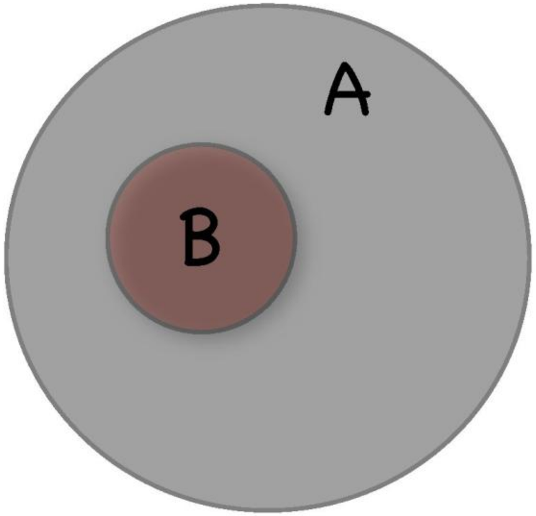
Entanglement entropy

Consider a system in a quantum state $|\psi\rangle$ ($\rho=|\psi\rangle\langle\psi|$)



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Alice can measure only in A, while Bob in the remainder B

Alice measures are entangled with Bob's ones: Schmidt deco

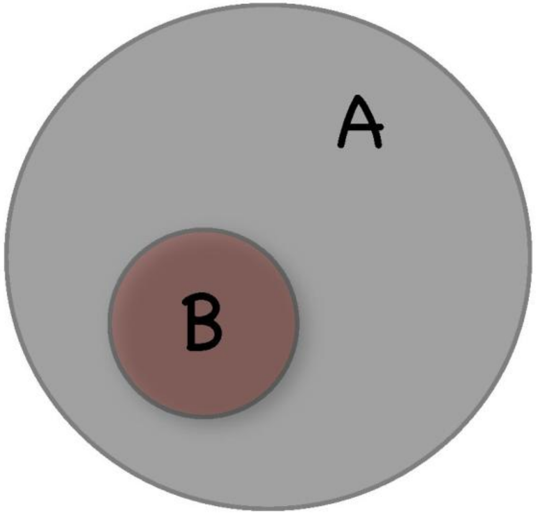
$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle_A |\Psi_n\rangle_B \qquad c_n \geq 0, \ \sum_n c_n^2 = 1$$

- If $c_1=1 \Rightarrow |\psi\rangle$ unentagled
- If c_i all equal $\Rightarrow |\psi\rangle$ maximally entangled

A natural measure is the entanglement entropy ($\rho_A = \mathrm{Tr}_B\, \rho$)

$$S_A \equiv -\mathrm{Tr}\, \rho_A \ln \rho_A = -\sum_n c_n^2 \ln c_n^2 = S_B$$

Entanglement entropy

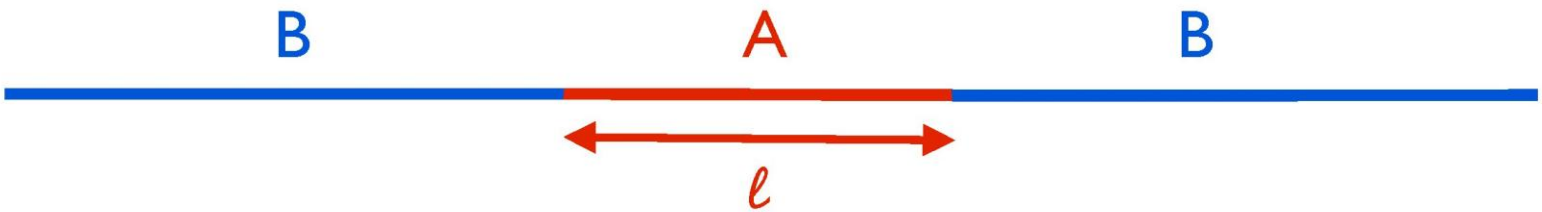


If $|\psi\rangle$ is the ground state of a local Hamiltonian

Area Law

$S_A \propto$ Area separating A and B [Srednicki '93]

If the Hamiltonian has a gap



In a 1+1 D CFT Holzhey, Larsen, Wilczek '94

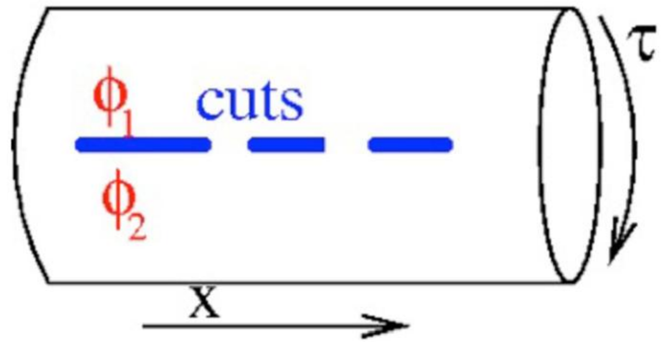
$$S_A = \frac{c}{3} \ln \ell$$

This is the most effective way to determine the central charge

Path integral and Riemann surfaces

[PC, Cardy 04]

$\langle \phi_1(x) | \rho_A | \phi_2(x) \rangle$ =



$\mathrm{Tr}\, \rho_A^n =$

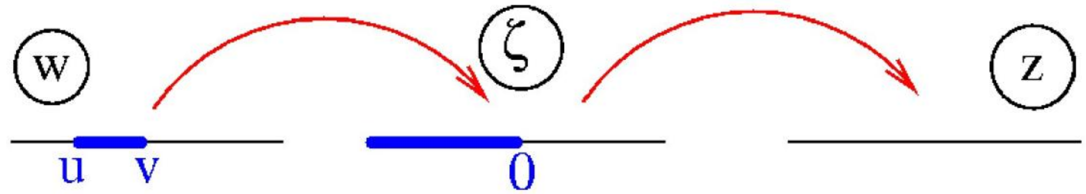
For n integer, $\mathrm{Tr}\,\rho_A^n$ is the partition function on a n-sheeted Riemann surface

Replica trick: $S_A = - \lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$

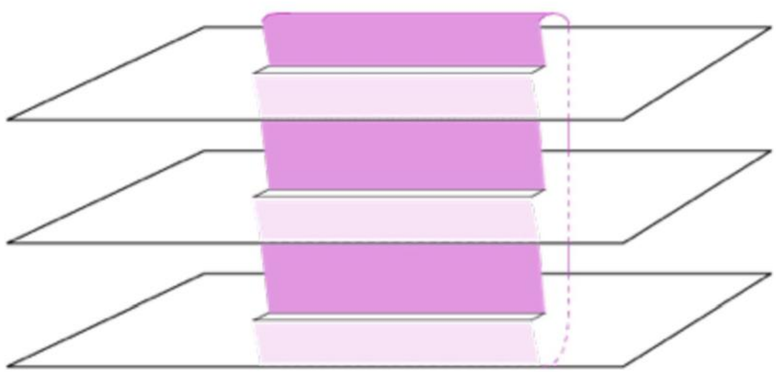
Riemann surfaces and CFT

This Riemann surface is mapped to the plane by

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



$\mathrm{Tr}\,\rho_A^n =$



$= c_n |u-v|^{-\frac{c}{6}(n-1/n)}$

$|u\text{-}v| = \ell$

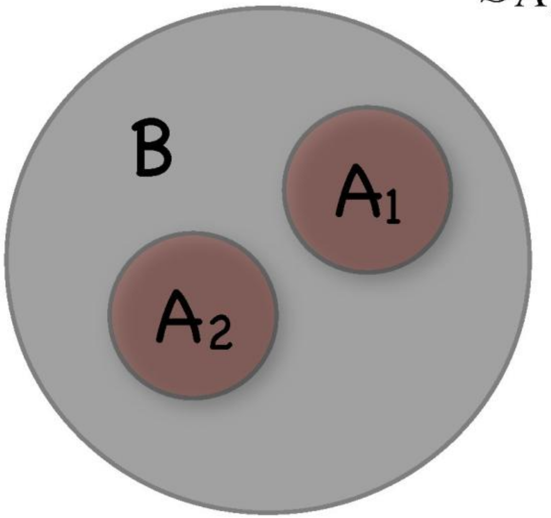
$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = \frac{c}{3}\log \ell$$

$\mathrm{Tr}\,\rho_A^n$ is equivalent to the 2-point function of twist fields

$\mathrm{Tr}\rho_A^n = \langle \mathcal{T}_n(u)\,\bar{\mathcal{T}}_n(v)\rangle$ with scaling dimension $\Delta_{\mathcal{T}_n} = \frac{c}{12}\left(n - \frac{1}{n}\right)$

Entanglement of non-complementary parts

$S_{A_1 \cup A_2}$ gives the entanglement between A and B



The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound on the entanglement between A_1 and A_2

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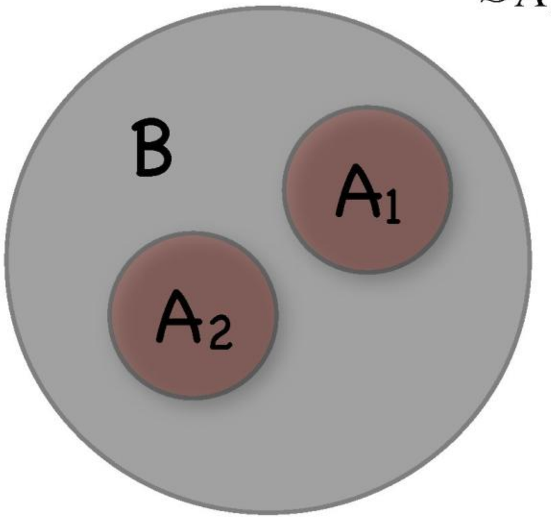
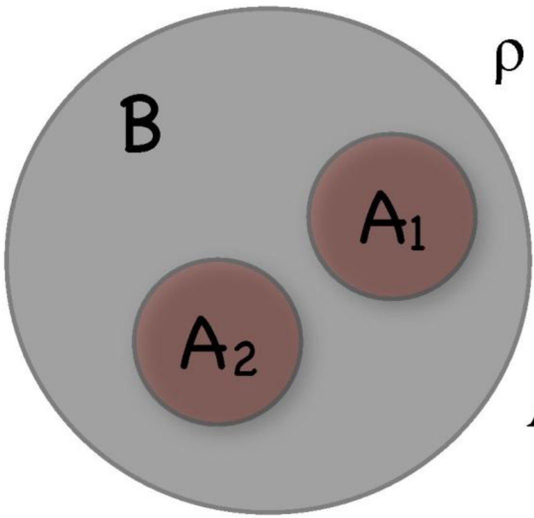


The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound on the entanglement between A_1 and A_2

What is the entanglement between the two non-complementary parts A_1 and A_2?

A **computable** measure of entanglement exists:
the logarithmic negativity [Vidal-Werner 02]

Entanglement negativity



Let us denote with $|e_i^{(1)}\rangle$ and $|e_j^{(2)}\rangle$ two bases in A_1 and A_2

ρ is the density matrix of $A_1 \cup A_2$, not pure

The **partial transpose** is

$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle$$

And the **logarithmic negativity**

$$\mathcal{E} \equiv \ln \| \rho^{T_2} \| = \ln \mathrm{Tr}\, |\rho^{T_2}|$$

$$\mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = \sum_{\lambda_i > 0} \lambda_i - \sum_{\lambda_i < 0} \lambda_i$$

It measures "how much" the eigenvalues of ρ^{T_2} are negative because $\mathrm{Tr}(\rho^{T_2})=1$

$\mathcal{E}$ is an **entanglement monotone** (does not decrease under LOCC)

It is also additive

A replica approach

- Let us consider traces of integer powers of ρ^{T_2}

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e} \quad n_e \text{ even}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o} \quad n_o \text{ odd}$$

- The analytic continuations from n_e and n_o are different

$$\mathcal{E} = \lim_{n_e \to 1} \ln \mathrm{Tr}(\rho^{T_2})^{n_e} \qquad \lim_{n_o \to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\rho^{T_2} = 1$$

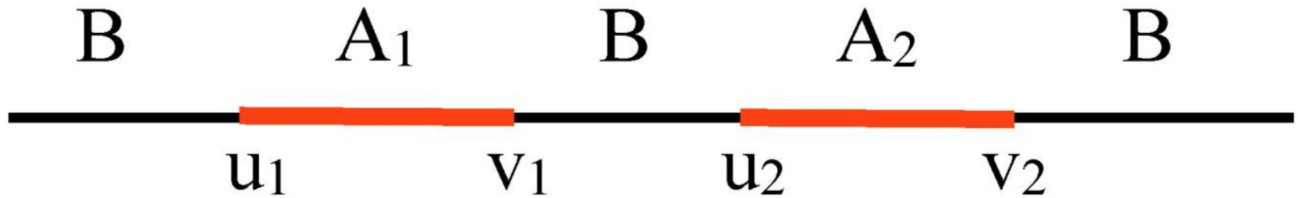
- For a pure state $\rho=|\psi\rangle\langle\psi|$ $\mathrm{Tr}\,(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \text{ odd} \\ (\mathrm{Tr}\,\rho_2^{n/2})^2 & n = n_e \text{ even} \end{cases}$

  - For $n_e \to 1$, we recover

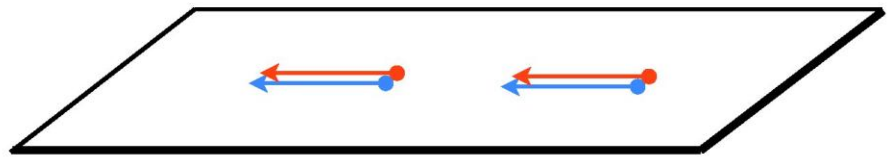
$$\mathcal{E} = 2 \ln \mathrm{Tr}\rho_2^{1/2} \quad \text{Renyi entropy } 1/2$$

Negativity and QFT

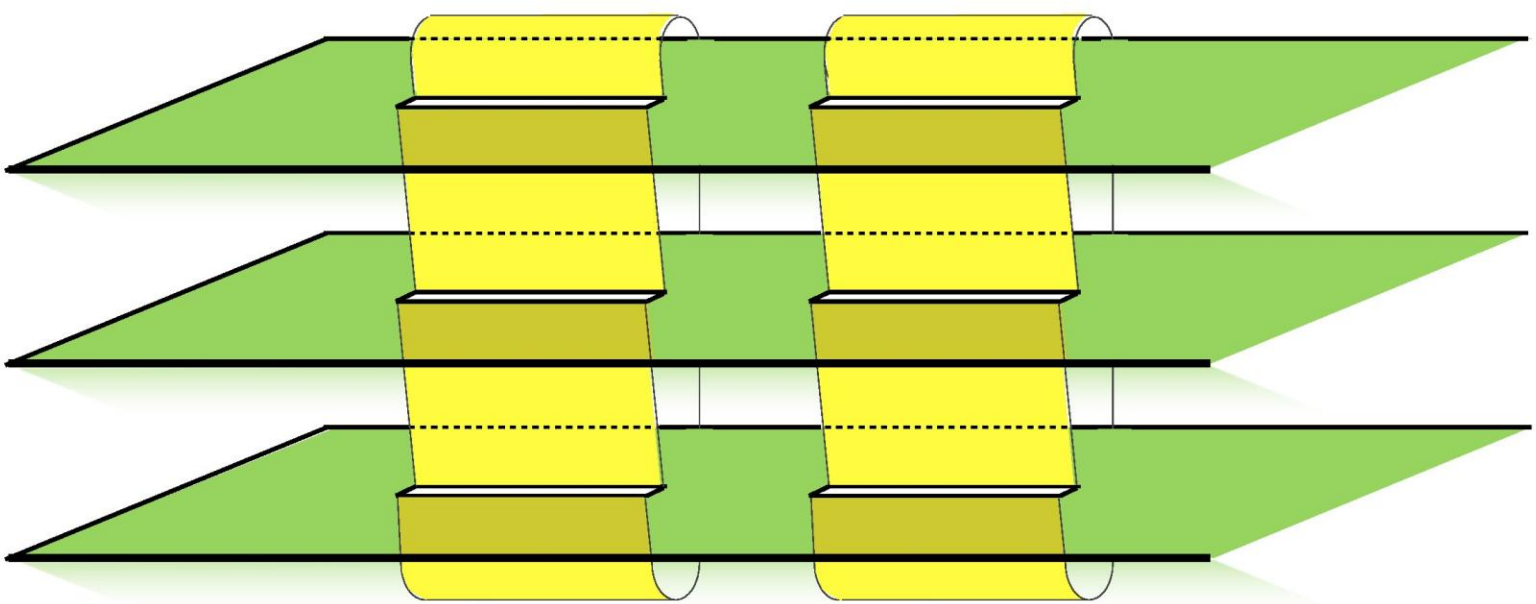
Tripartion:



ρ_A=

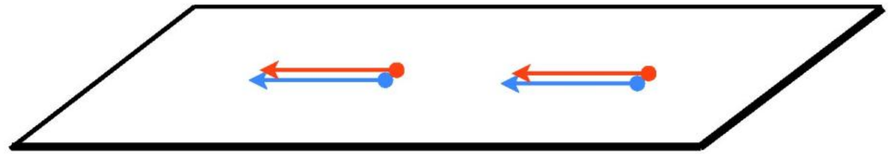


$$\mathrm{Tr}\rho_A^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$



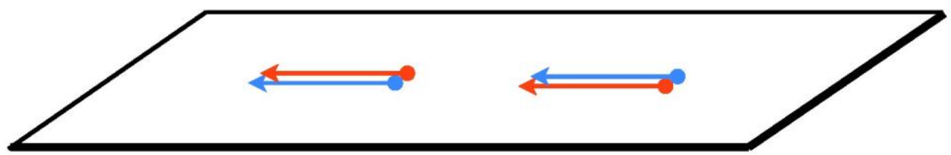
Negativity and QFT

$\rho_A =$

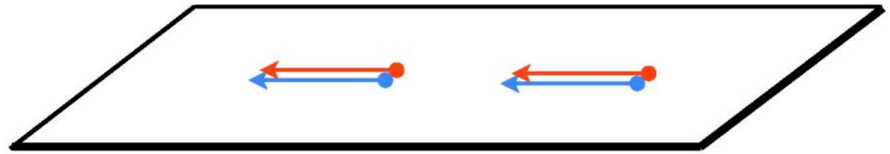


The partial transposition with respect to A_2 corresponds to exchange row and column indices in A_2

$\rho_A^{T_2} =$



Negativity and QFT

$\rho_A =$ 

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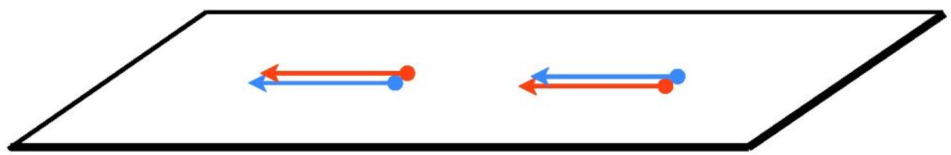
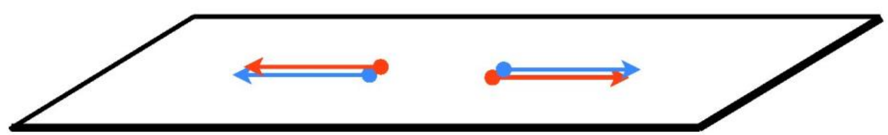
$\rho_A^{T_2} =$ 

It is convenient to reverse the order of indices

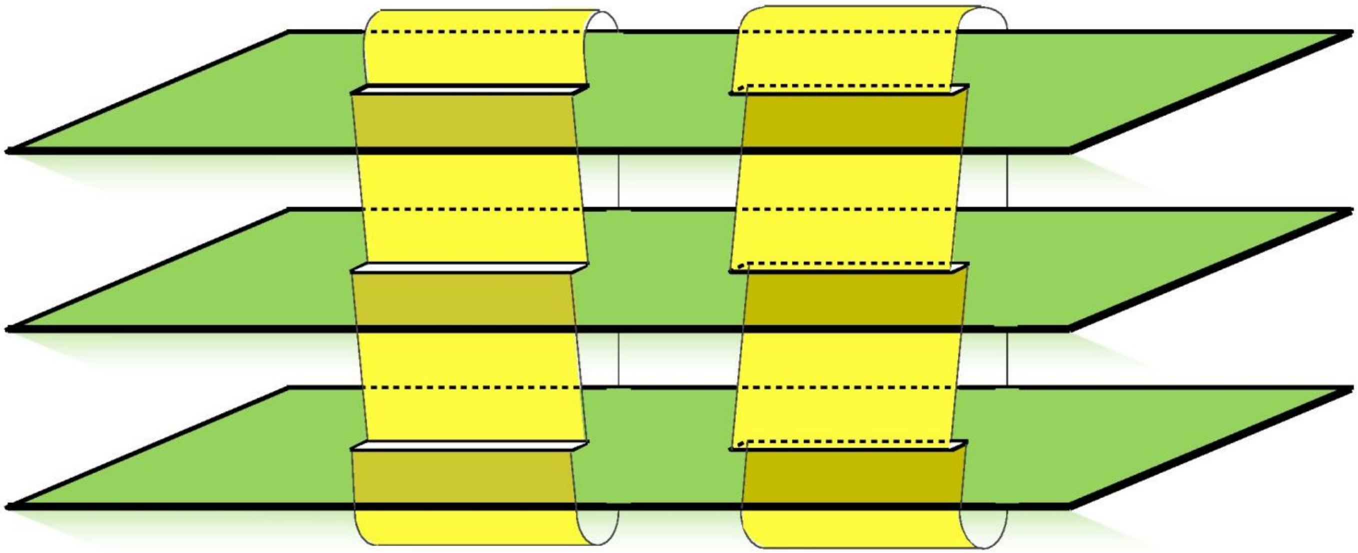
$\rho_A^{C_2} = C \rho_A^{T_2} C =$ 

$$\mathrm{Tr}(\rho_A^{T_2})^n = \mathrm{Tr}(\rho_A^{C_2})^n$$

Negativity and QFT

Gluing together *n* of the above

$$\mathrm{Tr}(\rho_A^{T_2})^n =$$

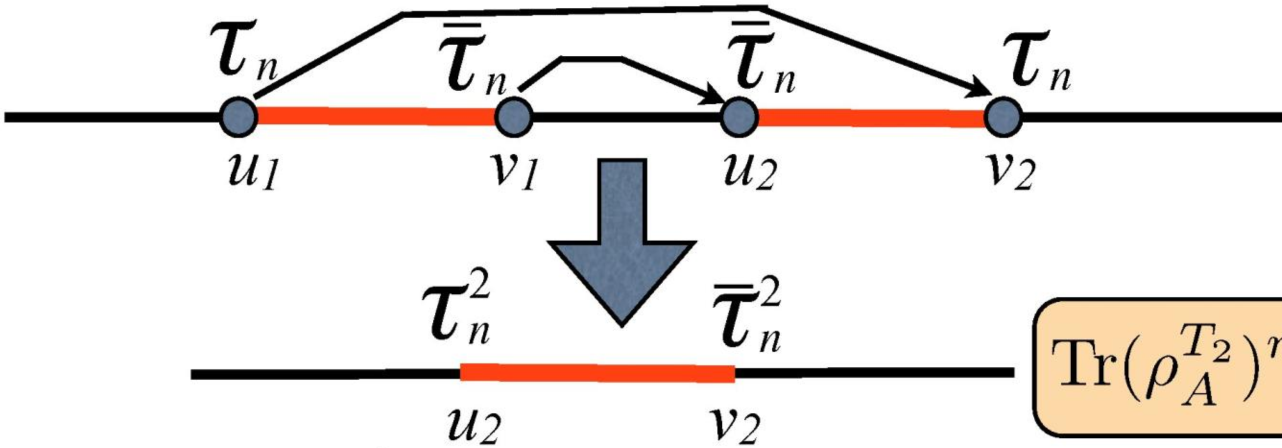


$$= \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

The partial transposition exchanges two twist operators

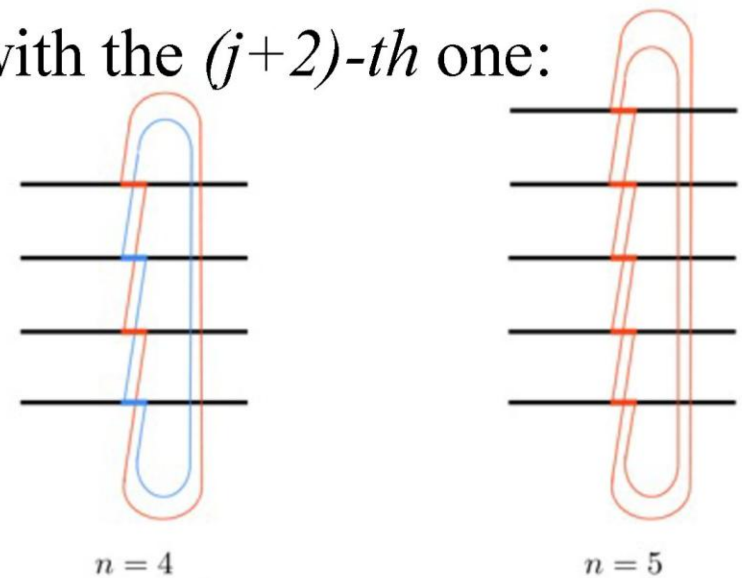
$\mathrm{Tr}(\rho_A \rho_A^{T_2})$ is the partition function on a Klein bottle

Pure States in QFT



$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2)\bar{\mathcal{T}}_n^2(v_2)\rangle$$

$\mathcal{T}_n^2$ connects the *j-th* sheet with the *(j+2)-th* one:



- For $n=n_e$ even, the R-surface decouples in two $n_e/2$ surface
- For $n=n_o$ odd, the n_o-sheeted surface remains n_o-sheeted

$$\begin{aligned}
\mathrm{Tr}(\rho_A^{T_2})^{n_e} &= (\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle)^2 = (\mathrm{Tr}\rho_{A_2}^{n_e/2})^2 \\
\mathrm{Tr}(\rho_A^{T_2})^{n_o} &= \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\rho_{A_2}^{n_o} ,
\end{aligned}$$

Pure States in CFT

From $\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2)\bar{\mathcal{T}}_n^2(v_2)\rangle$ and

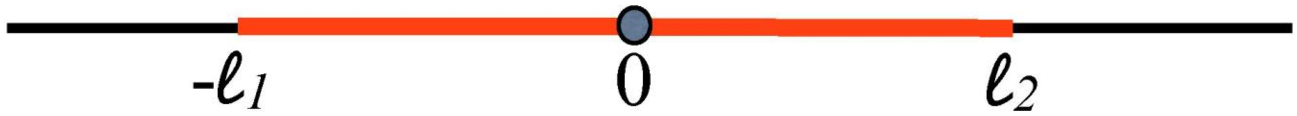
$$\begin{aligned}\mathrm{Tr}(\rho_A^{T_2})^{n_e} &= (\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle)^2 = (\mathrm{Tr}\rho_{A_2}^{n_e/2})^2 \\ \mathrm{Tr}(\rho_A^{T_2})^{n_o} &= \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\rho_{A_2}^{n_o} \,,\end{aligned}$$

$\mathcal{T}_{n_o}^2$ has dimension $\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right)$, the same as $\mathcal{T}_{n_o}$

$\mathcal{T}_{n_e}^2$ has dimension $\boxed{\Delta_{\mathcal{T}_{n_e}^2} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)}$

$$||\rho_A^{T_2}|| = \lim_{n_e \to 1} \mathrm{Tr}(\rho_A^{T_2})^{n_e} \propto \ell^{\frac{c}{2}} \Rightarrow \mathcal{E} = \frac{c}{2}\ln \ell + \mathrm{cnst}$$

Two adjacent intervals



3-point function:

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} \propto (\ell_1\ell_2)^{-\frac{c}{6}(\frac{n_e}{2}-\frac{2}{n_e})}(\ell_1+\ell_2)^{-\frac{c}{6}(\frac{n_e}{2}+\frac{1}{n_e})}$$

$$||\rho_A^{T_2}|| \propto \left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right)^{\frac{c}{4}} \Rightarrow \boxed{\mathcal{E} = \frac{c}{4}\ln\frac{\ell_1\ell_2}{\ell_1+\ell_2} + \mathrm{cnst}}$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} \propto (\ell_1\ell_2(\ell_1+\ell_2))^{-\frac{c}{12}(n_o-\frac{1}{n_o})}$$

Two disjoint intervals

[PC, Cardy Tonni 09/11]
[Furukawa et al 09]
[Caraglio, Gliozzi 09]

Prelude: The entanglement entropy

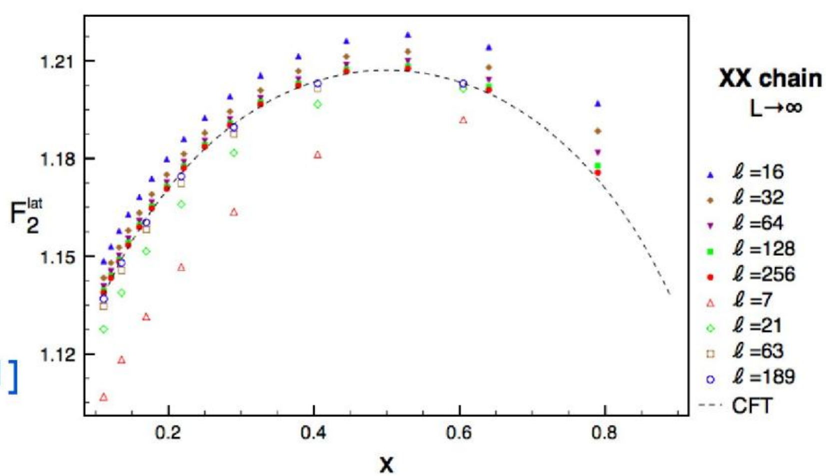
$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x)$$

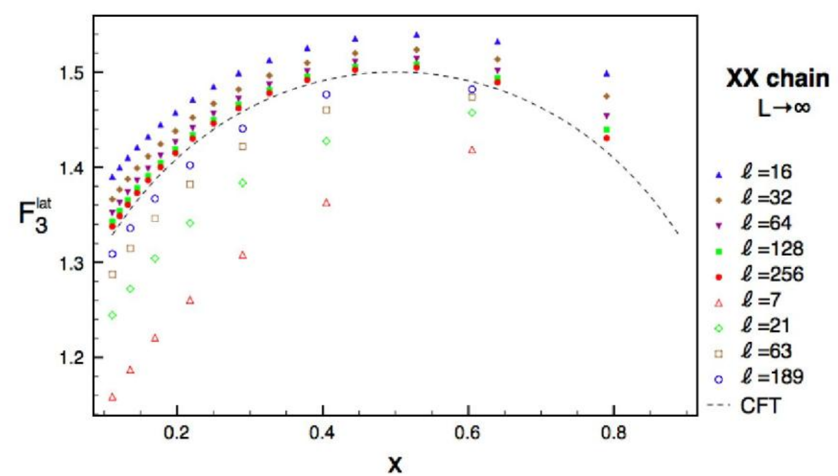
$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)}$$

4-point ratio

$F_n(x)$ is a calculable function depending on the full operator content

E.g. for Luttinger CFT:
$$F_n(x) = \frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}$$





[PC, Fagotti 11]

Two disjoint intervals

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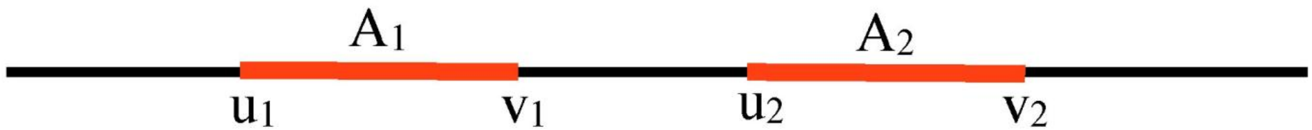
Prelude: The entanglement entropy



$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1-u_2||v_1-v_2|}{|u_1-v_1||u_2-v_2||u_1-v_2||u_2-v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x) \qquad x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)} = 4-\text{point ratio}$$

$F_n(x)$ is a calculable function depending on the full operator content

It admits the universal expansion

$$\mathrm{Tr}\,\rho_A^n = c_n^2(\ell_1\ell_2)^{-\frac{c}{6}(n-\frac{1}{n})} \sum_{\{k_j\}} \left(\frac{\ell_1\ell_2}{n^2r^2}\right)^{\sum_j(\Delta_j+\overline{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j}\left(e^{2\pi i j/n}\right)\rangle_{\mathsf{C}}^2$$

Trivial, but important for the following: At $n=1$ all coefficients are vanishing, since $\mathrm{Tr}\,\rho_A=1$

Two disjoint intervals

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

$$\mathrm{Tr}(\rho_A^{T_2})^n \propto [\ell_1 \ell_2 (1-y)]^{-\frac{c}{6}(n-\frac{1}{n})}\mathcal{G}_n(y)$$

Being $\mathrm{Tr}\rho_A^n$ and $\mathrm{Tr}(\rho_A^{T_2})^n$ related by an exchange of twists:

$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}(n-\frac{1}{n})}\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

$$\mathcal{E}(y) = \lim_{n_e \to 1} \ln \mathcal{G}_{n_e}(y) = \lim_{n_e \to 1} \ln \left[\mathcal{F}_{n_e}\left(\frac{y}{y-1}\right)\right]$$

Two disjoint intervals

$$\mathcal{E}(y) = \lim_{n_e \to 1} \ln \mathcal{G}_{n_e}(y) = \lim_{n_e \to 1} \ln \left[\mathcal{F}_{n_e} \left(\frac{y}{y-1} \right) \right]$$

Consequences:

- The Negativity is a scale invariant quantity!
- Since $\mathcal{F}_n(y) = \sum_i y^{2\Delta_i} s_n(i)$, $\mathcal{E}(y)$ vanishes in $y=0$ faster than any power
- For $u_1 \to v_2$, $y \to 1$ and we recover the result for adjacent intervals

$\mathcal{G}(y) \to - c/4 \ln(1-y)$ times possible log corrections

i.e. the negativity diverges for $y \to 1$

Finite Systems

A finite system of length L with PBC can be obtained mapping the the plane to the cylinder with the conformal mapping

$$z \rightarrow w = \frac{L}{2\pi} \log z$$

This has the net effect to replace any length with

$$\ell \rightarrow \frac{L}{\pi} \sin \frac{\pi \ell}{L}$$

Thus for two adjacent intervals we have

$$\mathcal{E}(y) = \frac{c}{4} \ln \left(\frac{L}{\pi} \frac{\sin(\frac{\pi \ell_1}{L}) \sin(\frac{\pi \ell_2}{L})}{\sin \frac{\pi(\ell_1 + \ell_2)}{L}} \right) + \text{cnst}$$

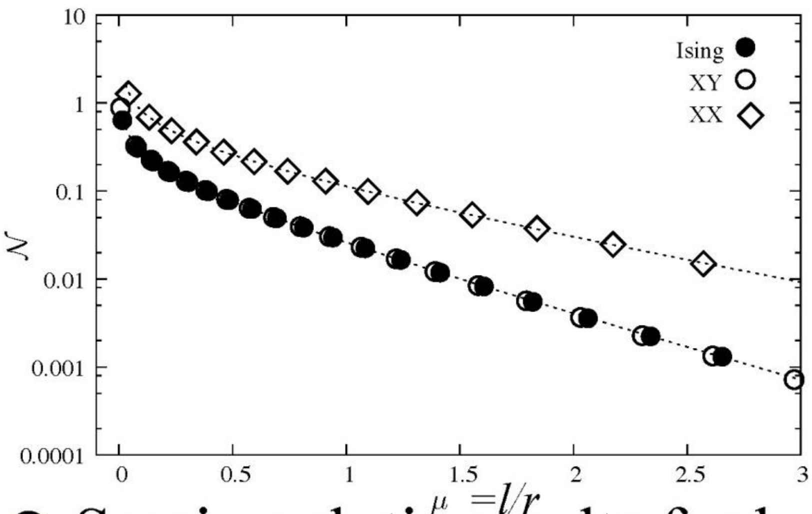
while for two disjoint ones of the same length l at distance r

$$\mathcal{E}(y) = \lim_{n_e \to 1} \ln \mathcal{G}_{n_e}(y) \quad \text{with} \qquad y = \left(\frac{\sin \pi \ell / L}{\sin \pi (\ell + r)/L} \right)^2$$

Numerical data: previous results

- DMRG results for Ising and XX chain. Two disjoint intervals

Wichterich et al



Proposed scaling: $\mathcal{N}(\rho_{SE}) \sim \mu^{-h}\, e^{-\alpha\mu}$

α=0.96, h=0.47 XX

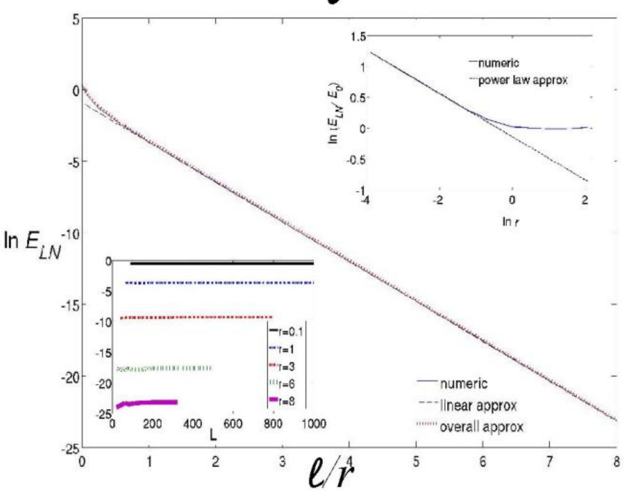
α=1.68, h=0.38 Ising

Good exponential, bad power law

Fit unstable

- Semi-analytic results for harmonic chain. Two disjoint intervals

Marcovitch et al



Proposed scaling:

$$E_{LN}^{\text{critical}} \sim \left(a r^{-\alpha} + f(r)\right) e^{-\beta_c r} \qquad \text{r}=\ell/r$$

α~1/3

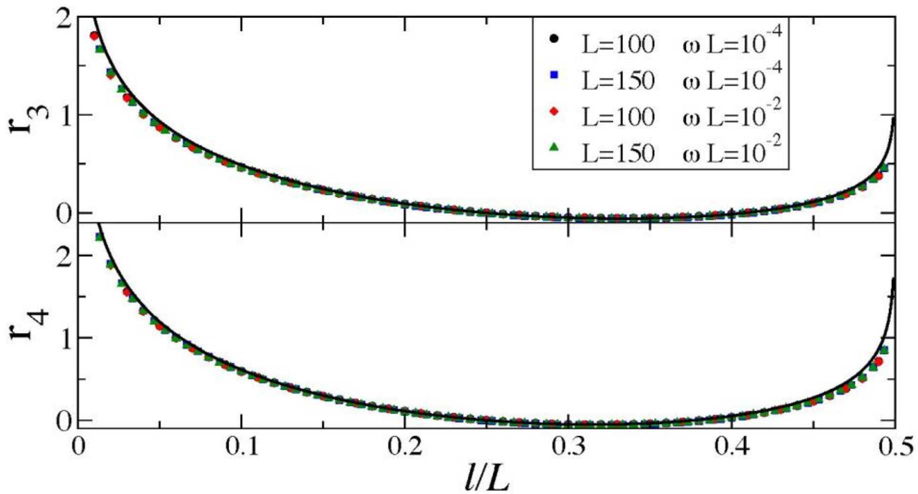
Good exponential, bad power law

Numerical data: new results

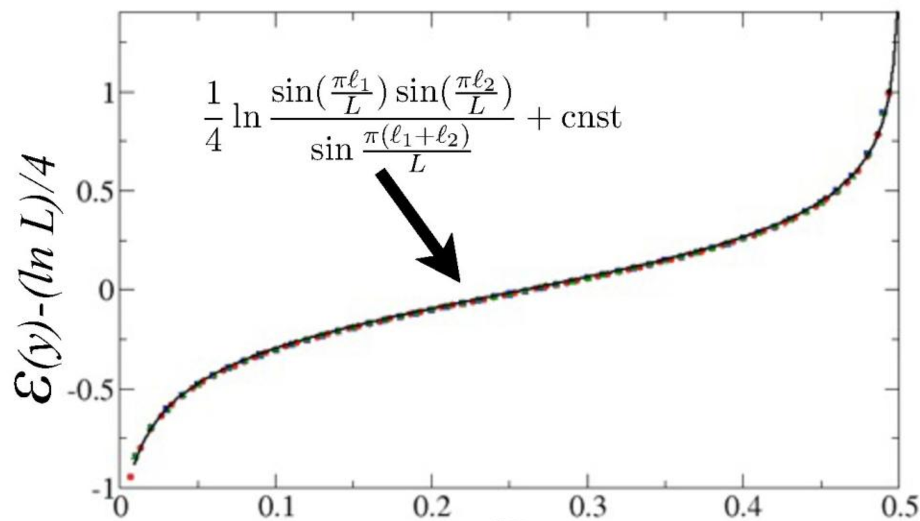
Semi-analytic results for harmonic chain

$$H = \frac{1}{2}\sum_{j=1}^{L}\left[p_j^2 + \omega^2 q_j^2 + \left(q_{j+1} - q_j\right)^2\right]$$ critical for ω=0

Two adjacent intervals of length ℓ:

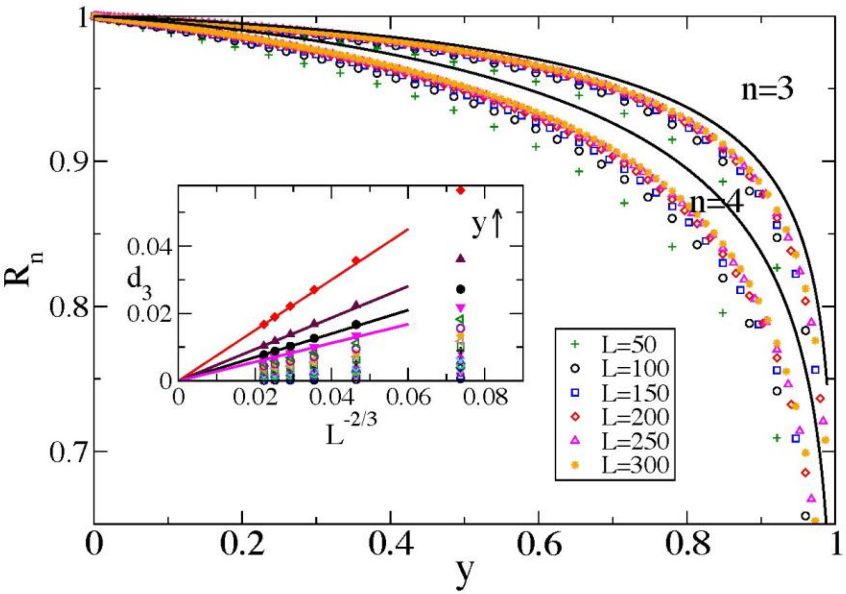


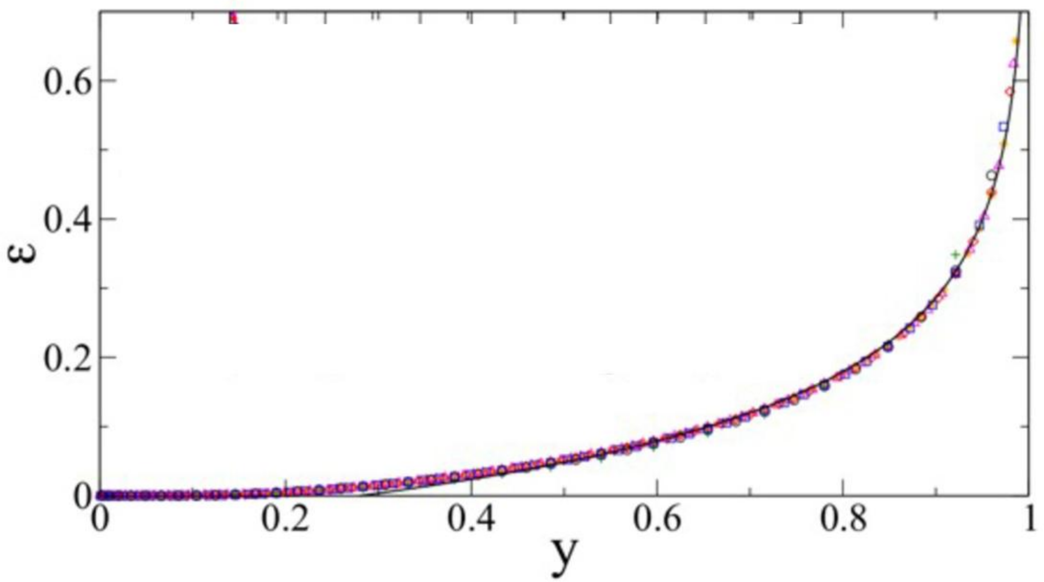
$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$



Numerical data: new results

Two disjoint intervals of length ℓ:





$$R_n(y) \equiv \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\rho_A^n}$$

$$R_n^{\mathrm{CFT}}(y) = \left[\frac{(1-y)^{\frac{2}{3}(n-\frac{1}{n})} \prod_{k=1}^{n-1} F_{\frac{k}{n}}(y) F_{\frac{k}{n}}(1-y)}{\prod_{k=1}^{n-1} \mathrm{Re}\big(F_{\frac{k}{n}}(\frac{y}{y-1}) \bar{F}_{\frac{k}{n}}(\frac{1}{1-y})\big)} \right]^{\frac{1}{2}}$$

$\mathscr{E}(y) \to -1/4 \ln(1-y) + 1/2 \ln(-\ln(1-y))$

Problem:

No analytic continuation

Open problems

- Work out the analytic continuation at $n_e \to 1$
  at least in some limiting cases
  (even for the entanglement entropy)

- An approach for calculating the negativity for free fermions is still missing!

- Out of equilibrium?