Expectation Values in Eigenstates of the Reduced Density Matrix

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Motivation

- DMRG allows for very accurate estimates of ground state energies in many-body systems
- it works by considering a finite region of length ℓ and truncating the spectrum of the reduced density matrix (rather than the spectrum of the hamiltonian), for progressively larger values of ℓ
- it is less effective at measuring correlation functions in the ground state since the truncation necessarily breaks translational invariance and there are large effects near the boundaries (one-point functions can be obtained by differentiating the ground state energy wrt a coupling constant)
- for the purpose of confronting a given analytic theory with numerical results it might be better to derive analytic results for correlations in the truncated space, or even better, in particular eigenstates of the density matrix

In this talk I will present some of these analytic results, for

- critical theories (CFT) in 1+1 dimensions
- integrable gapped theories in 1+1 dimensions
- some limited results in higher dimensions

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- en route I will summarise the path integral approach to computing entanglement entropy

Basic setup

- quantum system, Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- $A =$ degrees of freedom in some subregion, $B =$ the rest
- whole system is in the ground state $|0\rangle$ of some local hamiltonian H
- reduced density matrix $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} |0\rangle\langle 0|$
- ρ_A has eigenvalues $\{\lambda\}$ and eigenstates $|\lambda\rangle \in \mathcal{H}_A$ with $\lambda \geq 0$ and $\sum \lambda = 1$
- entanglement entropy

$$S = -\mathrm{Tr}\, \rho_A \log \rho_A = -\sum \lambda \log \lambda = \lim_{n \to 1} (1-n)^{-1} \log \mathrm{Tr}\, \rho_A^n$$

Density of states

$$\mathrm{Tr}\,\rho_A^n = \sum_\lambda \lambda^n = \int_0^{\lambda_{\max}} p(\lambda)\lambda^n d\lambda$$

where $p(\lambda)$ is the density of eigenstates of ρ_A.

- as $n \to \infty$, $\mathrm{Tr}\,\rho_A^n \sim \lambda_{\max}^n$
- writing $\lambda = \lambda_{\max} e^{-u}$ this has the form of a Laplace transform and so

$$\lambda p(\lambda) = \int_{c-i\infty}^{c+i\infty} e^{nu} \left(\mathrm{Tr}\,\rho_A^n\right) \frac{dn}{2\pi i}$$

- if $\mathcal{O} = \prod_i \Phi_i(x_i)$ is some product of local observables with $x_i \in A$, can we say something about

$$\langle \lambda | \mathcal{O} | \lambda \rangle \quad \text{and} \quad \langle \lambda | \mathcal{O} | \lambda' \rangle \quad \text{with } \lambda \neq \lambda' ?$$

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$$\mathrm{Tr}\left(\mathcal{O}\rho_A^n\right) = \int_0^{\lambda_{\max}} \overline{\langle\lambda|\mathcal{O}|\lambda\rangle} p(\lambda)\lambda^n d\lambda$$

$$\mathrm{Tr}\left(\mathcal{O}\rho_A^{n_1}\mathcal{O}\rho_A^{n_2}\right) = \int_0^{\lambda_{\max}}\int_0^{\lambda_{\max}} \overline{|\langle\lambda_1|\mathcal{O}|\lambda_2\rangle|^2} p(\lambda_1)p(\lambda_2)\lambda_1^{n_1}\lambda_2^{n_2} d\lambda_1 d\lambda_2$$

where $\overline{\langle\ldots\rangle}$ is the average over states in $(\lambda, \lambda + d\lambda)$

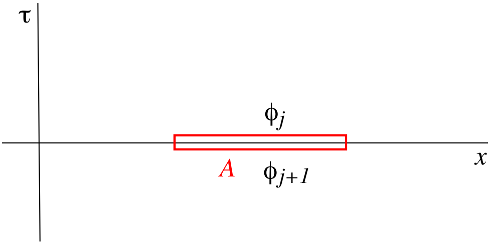
- if we can compute the lhs we can get information about these averages and in particular expectation values in $|\lambda_{\max}\rangle$

Path integral formulation

- in some basis $|\{\phi(x)\}\rangle$, the ground state wave functional $\Psi[\{\phi(x)\}] = \langle\{\phi(x)\}|0\rangle$ is given by the path integral in imaginary time $-\infty < \tau \leq 0$ conditioned on the values on $\tau = 0$

$$\Psi[\{\phi(x)\}] = Z^{-1/2} \int_{\{\phi(x,0)\}=\{\phi(x)\}} \mathcal{D}\{\phi(x,\tau)\}\, e^{-\int_{-\infty}^{0} L[\phi] d\tau}$$

- similarly for Ψ^*, from $0 \leq \tau < +\infty$
- $\langle\{\phi(x)\}|\rho_A|\{\phi(x)'\}\rangle$ is given by the path integral over $(-\infty < \tau < +\infty) \times \{\text{space}\}$ slit open along $A \times \{\tau = 0\}$, conditioned on the values $(\{\phi\}, \{\phi'\})$ above and below the slit



$$\mathrm{Tr}\,\rho_A^n = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_n(\langle\phi_1|\rho_A|\phi_2\rangle \ldots \langle\phi_n|\rho_A|\phi_1\rangle) = \frac{Z_{\mathcal{C}_n}}{Z^n}$$

where $Z_{\mathcal{C}_n}$ is the partition function on an n-sheeted cover of space-imaginary time, with conical singularities on $\partial A \cap \{\tau = 0\}$

Density of states

- Calabrese and Lefevre [2008] observed that in several simple cases for critical and near-critical systems in 1+1 dimensions
$$\mathrm{Tr}\,\rho_A^n \sim c_n e^{-b(n-1/n)}$$
where $b \propto c \log(L_{\rm eff}/a)$
- note that by taking $n \to \infty$, $b = -\log \lambda_{\rm max}$
- c_n is not determined in general, but $c_1 = 1$ and we can set $c_\infty = 1$ by adjusting a. CL observed that $c_n \approx 1$ in soluble examples. Assuming $c_n \equiv 1$ they inverted the Laplace transform to get

$$p(\lambda) = \delta(\lambda - \lambda_{\rm max}) + \frac{b\theta(\lambda_{\rm max} - \lambda)}{\lambda\sqrt{b\log(\lambda_{\rm max}/\lambda)}} I_1(2\sqrt{b\log(\lambda_{\rm max}/\lambda)})$$

- as long as λ is not too close to $\lambda_{\max}$, the asymptotic behaviour, given by the saddle-point approximation, is enough:

$$\lambda p(\lambda) \sim \frac{1}{2\sqrt{\pi}} \left(\frac{b}{\log(\lambda_{\max}/\lambda)} \right)^{1/4} e^{\sqrt{b \log(\lambda_{\max}/\lambda)}}$$

- in the same spirit we will apply their method to $\mathrm{Tr}(\mathcal{O}\rho_A^n)$.

Example 1. $A =$ finite interval in 1+1-dimensional CFT

- suppose A is the interval (x_1, x_2) and Φ is a local scaling operator such that $\langle 0|\Phi(\zeta)\Phi(\zeta')|0\rangle \sim |\zeta - \zeta'|^{-2\Delta}$
- the mapping

$$z \to \zeta = \left(\frac{z - x_1}{x_2 - z}\right)^{1/n}$$

uniformizes $\mathcal{C}_n \to \mathbb{C}$ and

$$\langle \Phi(x)\Phi(x')\rangle_{\mathcal{C}_n} = \left|\frac{d\zeta}{dx}\right|^{\Delta} \left|\frac{d\zeta'}{dx'}\right|^{\Delta} |\zeta - \zeta'|^{-2\Delta}$$

- the final expression is rather complicated but it simplifies as $n \to \infty$:

$$\langle \lambda_{\max} | \Phi(x)\Phi(x') | \lambda_{\max} \rangle \sim \left[\frac{(x_2 - x_1)^2}{(x - x_1)(x_2 - x)(x' - x_1)(x_2 - x') \log\left(\frac{(x - x_1)(x_2 - x')}{(x_2 - x)(x' - x_1)}\right)} \right]^{\Delta}$$

- this behaves like $|x_1 - x_2|^{-2\Delta}$ in the middle of the interval but also shows the singular behaviour near the ends
- Δ can be extracted knowing only $|\lambda_{\max}\rangle$

Carrying out the inverse Laplace transform in the saddle-point approximation, we find, for $\lambda < \lambda_{\rm max}$

$$\overline{\langle\lambda|\Phi(x)\Phi(x')|\lambda\rangle} \sim e^{\sqrt{B\log(\lambda_{\rm max}/\lambda)}-\sqrt{b\log(\lambda_{\rm max}/\lambda)}}$$

where $B = b + \Delta\log[(x-x_1)(x_2-x')/(x_2-x_1)^2]$, $b = -\log\lambda_{\rm max}$

- note that close to the end points $B < 0$ and the behaviour becomes oscillatory as a function of x and x'

- taking the two operators on different replicas (in middle of interval)

$$\frac{\mathrm{Tr}_A\,\Phi(0)\rho_A^{n'}\Phi(0)\rho_A^{n-n'}}{\mathrm{Tr}_A\,\rho_A^n} \sim |x_1 - x_2|^{-2\Delta}\, n^{-2\Delta}\,\left|1 - e^{2\pi i n'/n}\right|^{-2\Delta}$$

- taking $n' \gg 1$ and $n \to \infty$

$$\int_{\lambda<\lambda_{\rm max}} p(\lambda)\overline{|\langle\lambda|\Phi(0)|\lambda_{\rm max}\rangle|^2}(\lambda/\lambda_{\rm max})^{n'}d\lambda \sim n'^{-2\Delta}$$

which tells us that, as $\lambda \to \lambda_{\rm max}$

$$p(\lambda)\overline{|\langle\lambda|\Phi(0)|\lambda_{\rm max}\rangle|^2} \propto (\lambda_{\rm max} - \lambda)^{2\Delta-1}$$

Excess of the hamiltonian density in an interval

- suppose $H=\sum_j h(x_j)\sim\int h(x)dx$ where $h(x)=T_{tt}(x)$
- on the full line, this is normalised so that $\langle h(x)\rangle=0$, but this is no longer the case in eigenstates of ρ_A

$$\frac{\int p(\lambda)\overline{\langle\lambda|h(x)|\lambda\rangle}\lambda^n d\lambda}{\int p(\lambda)\lambda^n d\lambda}=\langle T_{tt}(x)\rangle_{\mathcal{C}_n}=\frac{c(1-1/n^2)}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}$$

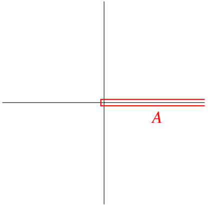
$$\text{So}\quad\langle\lambda_{\rm max}|h(x)|\lambda_{\rm max}\rangle=\frac{c}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}$$

- c may be extracted by truncating ρ_A to $|\lambda_{\rm max}\rangle$
- universal behaviour for $\lambda\leq\lambda_{\rm max}$

$$\overline{\langle\lambda|h(x)|\lambda\rangle}=\frac{c}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}\left(2-\frac{\log\lambda}{\log\lambda_{\rm max}}\right)$$

- changes sign at $\lambda\approx\lambda_{\rm max}^2$

Example 2. One-point functions in integrable models in 1+1-dimensions



- consider now a non-critical theory with correlation length ξ on the infinite line, with $A=(0,\infty)$. We can express ρ_A in terms of the corner transfer matrix (CTM): $\rho_A \propto (\mathrm{CTM})^4$

- for integrable models satisfying the Yang-Baxter relations the spectrum of ρ_A is of the form $q^{\Delta+N}$, where $q<1$ and the critical point is $q\to 1$. So we can write

$$\mathrm{Tr}\,\rho_A^n = \frac{Z(\mathcal{C}_n)}{Z^n} = \frac{\sum_j a_j\,\chi_{\Delta_j}(q^n)}{\left(\sum_j a_j\,\chi_{\Delta_j}(q)\right)^n}$$

where

$$\chi_{\Delta_j}(q) = q^{-c/24+\Delta_j}\sum_{N=0}^{\infty} d_N q^N$$

- the characters $\chi_{\Delta_j}(q)$ transform linearly under modular transformations $q \to \tilde{q}$ where $q = e^{-2\pi\delta}$, $\tilde{q} = e^{-2\pi i/\delta}$:

$$\chi_{\Delta_j}(q) = \sum_k S_{jk} \chi_{\Delta_k}(\tilde{q})$$

- as $q \to 1$, $\tilde{q} \propto \xi^{-2} \to 0$. The dominant term is the one with $\Delta_k = 0$, so

$$\mathrm{Tr}\,\rho_A^n \sim \frac{(\tilde{q}^{1/n})^{-c/24}}{(\tilde{q}^{-c/24})^n} \sim \xi^{-(c/12)(n-1/n)}$$

as expected

- one-point functions $\langle\Phi_k(0)\rangle$ are given by a similar formula with different coefficients a_j, so that the leading term as $\tilde{q} \to 0$ is

$$\langle\Phi_k(0)\rangle_{\mathcal{C}_n} \sim \frac{(\tilde{q}^{1/n})^{-c/24+\Delta_k}}{(\tilde{q}^{1/n})^{-c/24}} \sim \xi^{-2\Delta_k/n}$$

- from this we see, taking $n \to \infty$,

$$\langle\lambda_{\rm max}|\Phi_k(0)|\lambda_{\rm max}\rangle = O(1)$$

  - expectations of 1-point functions in $|\lambda_{\rm max}\rangle$ do not exhibit any critical behaviour!

- a more detailed analysis shows that, for $\lambda < \lambda_{\rm max}$,

$$\overline{\langle\lambda|\Phi(0)|\lambda\rangle} \sim \frac{\cos\left(\sqrt{(24\Delta/c-1)b\log(\lambda_{\rm max}/\lambda)}\right)}{e^{\sqrt{b\log(\lambda_{\rm max}/\lambda)}}}$$

  where $b = -\log\lambda_{\rm max} \sim (c/12)\log\xi$

Higher dimensions

- we can still relate $\mathrm{Tr}\,\rho_A^N$ to the partition function on $\mathcal{C}_n$, but conformal symmetry gives much less information
- in the case where A is the half-space $x_1 > 0$ and B is $x_1 < 0$, $\mathcal{C}_n = \{\text{2d conifold in } (x_0, x_1)\} \times \mathbb{R}^{d-2}$
- all the non-zero components of $\langle T_{\mu\nu}\rangle_{\mathcal{C}_n}$ are given in terms of one number a_n. In particular in cylindrical coordinates $(\rho, \theta, \vec{x}_\perp)$

$$\langle T_{\rho\rho}\rangle_{\mathcal{C}_n} = a_n/\rho^d$$

- for a free field theory in $d = 4$, $a_n \propto (1 - 1/n^4)$ but in general all we know is that $a_1 = 0$ and we expect a_∞ to be finite
- it has been argued that $a_1' \propto \mathfrak{a}$, the "$\mathfrak{a}$-anomaly" of the CFT which satisfies an $\mathfrak{a}$-theorem
- response to a scale transformation

$$\propto \int \langle T_{\rho\rho}\rangle_{\mathcal{C}_n}\, \rho d\rho d\theta d^{d-2}x_\perp \sim (R/a)^{d-2} L^{d-2} \quad \text{(area law)}$$

- this geometry may be conformally mapped so that A is the interior of a sphere S^{d-2} of radius R, and hence we can find $\langle T_{\mu\nu}\rangle$. Simplest case is the excess hamiltonian density inside the sphere

$$\langle T_{00}(r,0)\rangle_{\mathcal{C}_n} = a_n \left(\frac{2R}{R^2 - r^2}\right)^d$$

so, for example, a_∞ may be extracted from $|\lambda_{\max}\rangle$

- note that, close to the conical singularity, we now have, with $\rho = |R - r|$

$$\langle T\rangle \sim \frac{a_n}{\rho^d}\left(1 + O\left(\frac{\rho^2}{R^2}\right) + \cdots\right)$$

- this leads to a universal correction $\sim a_n \log(R/a)$ to the Rényi entropies for $d = 4$ (and other even d)

Summary

- in 1+1dimensions, critical exponents and the central charge c may in principle be extracted even if the reduced density matrix is truncated to a few (just one!) eigenstates
- in higher dimensions the analogue of c may also be extracted in simple geometries