Entanglement in gapless systems with a quantum impurity

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XXZ chain with quantum impurity

$$
\begin{aligned}
H &= J \sum_{i=-\infty}^{\infty} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) \\
&+ (J' - J) \left(S_0^x S_1^x + S_0^y S_1^y + \Delta S_0^z S_1^z \right)
\end{aligned}
$$

- Bulk chain in gapless Luttinger liquid phase for $-1 < \Delta \leq 1$
- One modified link ($J \to J'$) in the middle of the chain
  - For $J' > J$ existence of bound state
  - Consider henceforth weak-link case $0 < J' < J$
- Dimension of perturbation is $h = 2\left(1 + \pi^{-1} \arccos \Delta\right)$
  - Marginal for $\Delta = 0$: equivalent to Ising model with defect line [Oshikawa-Affleck 1996]
  - Relevant for $\Delta < 0$: healing RG-flow ($J' \to J$), $\xi_x \sim (J')^{1/(h-1)}$

Continuum limit

- At low energy: Right (R) and Left (L) movers
- Unfolding: Map formally $\mathrm{L} \to \mathrm{R}$ to get two chiral wires
- Form even/odd combinations: $\phi_{\mathrm{R}}^{\pm} = \frac{1}{\sqrt{2}} \left(\phi_{1\mathrm{R}}(x) \pm \phi_{2\mathrm{R}}(x) \right)$
- The odd one ϕ_{R}^{-} decouples from impurity, so forget it.
- The even one $\phi := \phi_{\mathrm{R}}^{+}$ satisfies

$$H = v_{\mathrm{F}} \int_{-\infty}^{\infty} \mathrm{d}x \, (\partial_x \phi)^2 + \lambda \cos \left(\beta \phi(0) \right)$$

  with $h = \beta^2 / 8\pi$.
- Can fold back to obtain boundary sine-Gordon model

Two weak links: tunneling through a resonant level

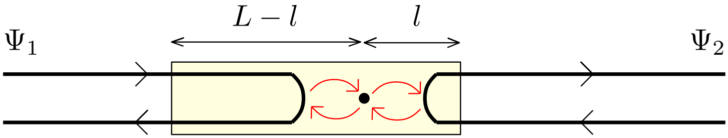
$$\begin{aligned} H &= J \sum_{i=-\infty}^{\infty} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) \\ &+ (J' - J) \sum_{i=-1}^{0} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) \end{aligned}$$

- Dimension of perturbation is now $h/2$ and is always relevant
- Healing length (Kondo temp.) $\xi_x := (T_B)^{-1} = (J')^{1/(h/2-1)}$
- Unfold and bosonise as before:

$$H = v_{\rm F} \int_{-\infty}^{\infty} {\rm d}x \, (\partial_x \phi)^2 + \lambda \left[{\rm e}^{i \frac{\beta}{\sqrt{2}} \phi} S^- + {\rm e}^{-i \frac{\beta}{\sqrt{2}} \phi} S^+ \right] (0)$$

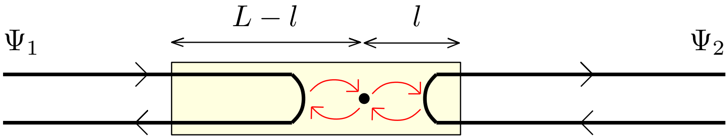
- Can fold back to obtain anisotropic Kondo problem

Entanglement entropy



The problem (here for $\Delta = 0$ only)

- What is the entanglement entropy $S_A = -\mathrm{Tr}_{\mathcal{H}_A} [\rho_A \log \rho_A]$ of an interval A of length L with the remainder of the system?
- Case $\alpha = \frac{1}{2}$ is easier [Saleur-Schmitteckert-Vasseur 2013]
    - 1^{st} order IR perturbation [Sørensen-Chang-Laflorencie-Affleck 2006]
- Consider in general the asymmetric case with $\ell = \alpha L$ and $\alpha \neq \frac{1}{2}$



$\ell = \alpha L$: The case $\alpha = \frac{1}{2}$ versus $\alpha \neq \frac{1}{2}$

- Case $\alpha = \frac{1}{2}$ can be mapped to a boundary problem by folding
  - Applies in general to any massless integrable system
  - But physically contrived: we really want $\alpha = 0$ (i.e. $A =$ one lead)!
- $\alpha \neq \frac{1}{2}$ requires defect scattering formalism (free systems only)

Limiting cases [Cardy-Calabrese 2004]

- UV limit $L \ll (T_B)^{-1}$:
  - Two half-chains and $S_A = \frac{c}{6} [\log(\ell) + \log(L - \ell)]$
- IR limit $L \gg (T_B)^{-1}$:
  - One bulk chain and $S_A = \frac{c}{3} \log(L)$

Twist fields [Cardy-Calabrese 2004]

- Replica trick: $S_A = -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \mathrm{Tr}_{\mathcal{H}_A} (\rho_A)^n$
- Continue analytically from $n \in \mathbb{N}$
- Hence define theory on multi-sheeted Riemann surface

- Branch-point twist fields $\mathcal{T}$ at $(x, y) = (a_1, 0)$ and $(a_2, 0)$

$$\mathrm{Tr}_{\mathcal{H}_A} (\rho_A)^n \propto \left\langle \mathcal{T}(a_1, 0) \tilde{\mathcal{T}}(a_2, 0) \right\rangle_{\mathcal{L}^{(n)}}$$

- Map to $z \in \mathbb{R}^2$ via $z = \left(\frac{w - a_1}{w - a_2} \right)^{1/n}$, obtaining $h_n = \bar{h}_n = \frac{c}{24} \left(n - \frac{1}{n} \right)$

Form Factor approach

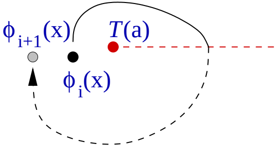
- We have a CFT in UV and IR, but in-between there is a flow
  - Hence we cannot directly use boundary CFT techniques
  - But we can use massless Form Factor (FF) approach [SSV 2013]

The problem can be attacked in several stages:

1. FF approach to twist field in massive bulk theory [Cardy - Castro-Alvaredo - Doyon 2007]
2. Map $\Delta = 0$ cases to Ising with boundary condition
3. FF approach for massive boundary Ising problem [Castro-Alvaredo - Doyon 2008]
4. For $\alpha = \frac{1}{2}$ take massless FF limit [SSV 2013]
5. For $\alpha \neq \frac{1}{2}$ use FF defect scattering formalism [Delfino-Mussardo-Simonetti 1994]

FF for branch-point twist fields [C-CA-D 2007]

- FF of local operator $\mathcal{O}$:
  $F_k^{\mathcal{O}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle^{\rm in}_{\mu_1,\ldots,\mu_k}$
- k particles with quantum numbers μ_i and rapidities θ_i
- Assume integrable theory with single particle spectrum and no bound states (Ising, sinh-Gordon)
- For twist field $\mathcal{T}$, the replicated S-matrix is $S_{ij}(\theta) = (S(\theta))^{\delta_{ij}}$
- $F_k^{\mathcal{T}|\ldots\mu_i\mu_{i+1}\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S_{\mu_i,\mu_{i+1}}(\theta_{i,i+1})F_k^{\mathcal{T}|\ldots\mu_{i+1}\mu_i\ldots}(\ldots,\theta_{i+1},\theta_i,\ldots)$
- $F_k^{\mathcal{T}|\mu_1\mu_2\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2,\ldots,\mu_n,\mu_1+1}(\theta_2,\ldots,\theta_k,\theta_1)$



- Further axioms for kinematic residue equations

Two-particle approximation [C-CA-D 2007]

- Insert complete set in $\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle$ and truncate to two particles:

$$\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \approx \langle \mathcal{T}\rangle^2 + \frac{1}{2!}\sum_{i,j=1}^{n}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_2}{2\pi}\left|F_2^{\mathcal{T}|ij}(\theta_{12},n)\right|^2 \mathrm{e}^{-rm(\cosh\theta_1+\cosh\theta_2)}$$

- Two-particle form factors $F_2^{\mathcal{T}|ij}(\theta_{12},n)$ given by

$$K(\theta) = \frac{F_2^{\mathcal{T}|11}}{\langle \mathcal{T}\rangle} = -i\frac{\cos\left(\frac{\pi}{2n}\right)\sinh\left(\frac{\theta}{2n}\right)}{n\sinh\left(\frac{i\pi+\theta}{2n}\right)\sinh\left(\frac{i\pi-\theta}{2n}\right)}$$

- Change to $\theta_1 \pm \theta_2$ variables and do one integral:

$$\begin{aligned}\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle &\approx \langle \mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\theta\, f(\theta,n)K_0(2rm\cosh(\theta/2))\right)\\ \langle \mathcal{T}\rangle^2 f(\theta,n) &= \sum_{j=1}^{n}\left|F_2^{\mathcal{T}|1j}(\theta,n)\right|^2\end{aligned}$$

- Analytic continuation $\tilde{f}(\theta,n)$ satisfies $\frac{\partial}{\partial n}\tilde{f}(\theta,n)\Big|_{n=1} = \pi^2\tilde{f}(1)\delta(\theta)$

- For the Ising model, $\tilde{f}(n) = 1/2$ for all n
- So the sub-leading contribution to S_A for $rm \gg 1$ is $-\frac{1}{8} K_0(2rm)$. This is universal.
- Neglected four-particle contributions are $O(\mathrm{e}^{-4rm})$

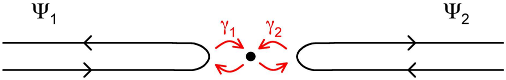
How to deal with the $m \to 0$ limit?

- Set $\frac{m}{2} = M\mathrm{e}^{-\theta_0}$ with $\theta_0 \to \infty$.
  Finite-energy excitations have $\theta = \pm(\theta_0 + \beta)$ with β finite.
  They are LR movers with $p = \pm M\mathrm{e}^{\beta}$ and $e = |p|$.
- Correction to S_A becomes: $-\frac{1}{8} \int_0^\infty \frac{\mathrm{d}\omega}{\omega} \mathrm{e}^{-2Mr\omega}$ (divergent at $\omega \ll 1$)
- Γ-function regularisation:
  $\int_0^\infty \mathrm{d}x \, x^{\eta-1} \mathrm{e}^{-2Mr\omega} = \frac{1}{(2Mr)^\eta} \Gamma(\eta) = \frac{1}{\eta} - \log(2Mr) + \ldots$
- Suppose we keep just the finite part: $S_A = \ldots + \frac{1}{8} \log(r) + \ldots$
- Higher-particle contributions will lead to $\frac{1}{6} \log(r) = \frac{c}{3} \log(r)$

Two weak links at $\Delta = 0$: Fermionic formulation

- Jordan-Wigner transformation of XX spin chain [Lieb-Schultz-Mattis 1961]

$$H = -J \sum_{i=-\infty}^{-2} c_{i+1}^\dagger c_i - J \sum_{i=1}^{\infty} c_{i+1}^\dagger c_i - J'(c_{-1}^\dagger c_0 + c_0^\dagger c_1) + \mathrm{h.c.}$$

- Continuum limit: 

$$H = \int_{-\infty}^{0} i\left(\psi_{1L}^\dagger \partial_x \psi_{1L} - \psi_{1R}^\dagger \partial_x \psi_{1R}\right) \mathrm{d}x + \int_{0}^{\infty} i\left(\psi_{2L}^\dagger \partial_x \psi_{2L} - \psi_{2R}^\dagger \partial_x \psi_{2R}\right) \mathrm{d}x + \lambda \left[\left(\psi_1^\dagger(0) + \psi_2^\dagger(0)\right) d + \mathrm{h.c.}\right]$$

- Unfold to get R movers only. Form $\Psi_R = \frac{1}{\sqrt{2}}(\psi_{1R} + \psi_{2R})$ (odd combination decouples). Refold (introducing Ψ_L):

$$H = -i \int_{-\infty}^{0} \left(\Psi_R^\dagger \partial_x \Psi_R - \Psi_L^\dagger \partial_x \Psi_L\right) \mathrm{d}x + \lambda\sqrt{2}\left[\Psi^\dagger(0) d + \mathrm{h.c.}\right]$$

- Decompose into $\Psi_R = \frac{1}{\sqrt{2}}(\xi_{1R} + i\xi_{2R})$ and $d = \frac{1}{\sqrt{2}}(d_2 + id_1)$.
- We get $H = H_1 + H_2$ for real fermions ($k = 1, 2$) with

$$H_k = -\frac{i}{2}\int_{-\infty}^{0} (\xi_{kR}\partial_x\xi_{kR} - \xi_{kL}\partial_x\xi_{kL})\,\mathrm{d}x + (-1)^{k-1}\frac{i}{\sqrt{2}}\lambda\xi_k(0)d_k$$

- Two independent Majorana fermions with boundary field $\sim \pm\lambda$
- Suggests computing S_A for $\alpha = \frac{1}{2}$ in boundary Ising model
- This was done in [SSV 2013] by extensive use of:
  - the FF results of [Castro-Alvaredo - Doyon 2008]
  - the boundary state of [Ghoshal-Zamolodchikov 1994]
- Does not work for $\alpha \neq \frac{1}{2}$, since folding is incompatible with the geometry of the interval A

FF computation: scattering on the defect

$$\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \approx \langle \mathcal{T}\rangle^2 + \frac{1}{2!}\sum_{i,j=1}^{n}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_2}{2\pi}\left|F_2^{\mathcal{T}|ij}(\theta_{12},n)\right|^2 \mathrm{e}^{-rm(\cosh\theta_1+\cosh\theta_2)}$$

- Instead we can now have two particles (RR or LL) created by $\mathcal{T}$, transmitted by the defect, and absorbed on $\tilde{\mathcal{T}}$:

$$\frac{1}{2}\sum_{i,j=1}^{n}\int\frac{\mathrm{d}\theta_1}{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\hat{T}(\theta_1)\hat{T}(\theta_2)\left|F_2^{\mathcal{T}|ij}(\theta_{12},n)\right|^2 \mathrm{e}^{-Lm(\cosh\theta_1+\cosh\theta_2)}$$

- Or two LR pairs created by $\mathcal{T}$ (or $\tilde{\mathcal{T}}$: formally $\ell \to L-\ell$) and reflected on the impurity:

$$\langle\mathcal{T}\rangle\sum_{i,j=1}^{n}\int\frac{\mathrm{d}\theta_1}{4\pi}\frac{\mathrm{d}\theta_2}{4\pi}\hat{R}(\theta_1)\hat{R}(\theta_2)F_4^{\mathcal{T}|iijj}(\theta_1,-\theta_1,\theta_2,-\theta_2,n)\,\mathrm{e}^{-2\ell m(\cosh\theta_1+\cosh\theta_2)}$$

- Other diagrams are forbidden by $\mathbb{Z}_2$ symmetry of the Ising model. It requires an even number of L and R. Note that indeed $F_2^{\mathcal{T}|ii}(\theta,-\theta)\to 0$ for $m\to 0$.

- After a calculation one finds, in the massless limit:

$$S_A(\ell, L) = -\frac{1}{4}\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\mathrm{e}^{-2L\omega}\,\hat{T}(\omega)^2 - \frac{1}{8}\int_0^\infty \frac{\mathrm{d}\omega}{\omega}\left[\mathrm{e}^{-4\ell\omega} + \mathrm{e}^{-4(L-\ell)\omega}\right]\hat{R}(\omega)^2$$

  - Needs Γ-function regularisation in the $\omega \ll 1$ limit
  - “Renormalise” by a factor $\frac{4}{3}$ to produce correct limiting values
- One weak link: $\hat{T}(\omega)^2 = \cos^2\xi$ and $\hat{R}(\omega)^2 = \sin^2\xi$
  - Here $\xi = \frac{\pi}{2} - 2\arctan(J'/J)$ independent of ω (marginal case)
- Two weak links: $\hat{T}(\omega)^2 = \left(\frac{T_B}{T_B+\omega}\right)^2$ and $\hat{R}(\omega)^2 = \left(\frac{\omega}{T_B+\omega}\right)^2$
  - Scaling function of $x = LT_B$, where $T_B = \frac{(J'/J)^2}{\sqrt{1-2(J'/J)^2}}$
  - Note that $\hat{R}^2 + \hat{T}^2 \neq 1$; the relation to reflexion / transmission probabilities involves a subtlety having to do with the choice of quantisation scheme.

Results for one weak link

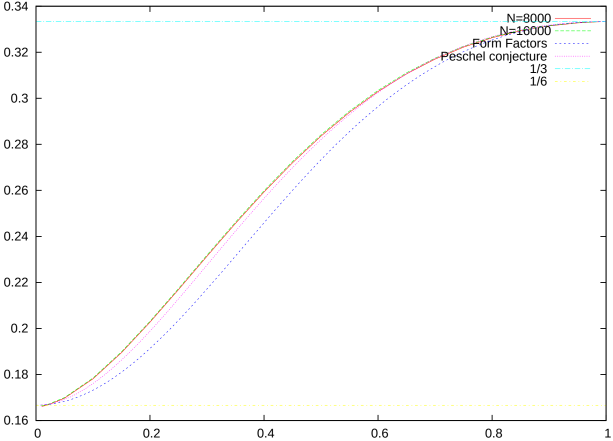
- Γ-regularise and renormalise to find:

$$S_A(\ell, L) = \frac{\cos^2 \xi}{3} \log L + \frac{\sin^2 \xi}{6} \left[\log \ell + \log(L - \ell)\right]$$

- Symmetric case ($\ell = L/2$): $S_A = \frac{1}{3} \log L$, independent of ξ
  - Check of $J' = J$ ($\xi = 0$) case: $\frac{c}{3} = \frac{1}{3}$
  - Check of $J' = 0$ ($\xi = \frac{\pi}{2}$) case: $2 \times \frac{c}{6} = \frac{1}{3}$
- Totally asymmetric case: $\ell = 0$ (or rather $\ell \sim$ lattice spacing)

$$S \sim \frac{1 + \cos^2 \xi}{6} \log L = \frac{1}{6} \left(1 + \frac{4(J'/J)^2}{\left(1 + (J'/J)^2\right)^2} \right) \log L$$

One weak link: comparison with numerics ($\alpha = 0$)



- Initial conjecture [Peschel 2005] slightly corrected by exact result [Peschel et al. 2008]

Results for two weak links

- Since there is a flow in $x = LT_B$, it is *not correct* to extract the apparent coefficient of $\log L$.
- Instead set $\ell = \alpha L$ and compute the scaling function

$$\frac{\partial S_A(\alpha)}{\partial \log L} = F(LT_B)$$

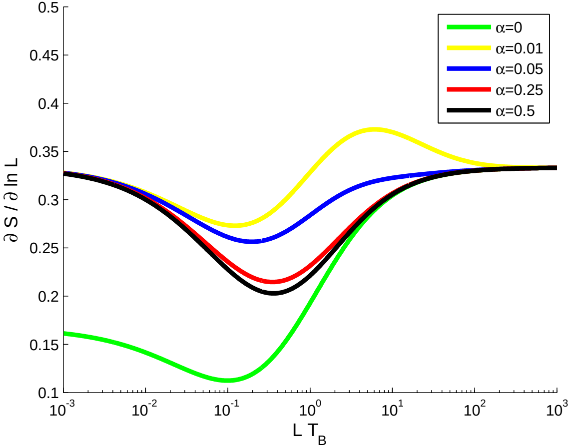
- After renormalisation we find:

$$F(x) = \frac{2}{3} \int_0^\infty \mathrm{d}v \, \mathrm{e}^{-2v} \left(\frac{x}{v+x} \right)^2 + \frac{2}{3} \int_0^\infty \mathrm{d}v \, \left(\alpha \mathrm{e}^{-4\alpha v} + (1-\alpha) \mathrm{e}^{-4(1-\alpha)v} \right) \left(\frac{v}{v+x} \right)^2$$

  - No need to Γ-regularise (the derivative does the job)
  - Case $\alpha = \frac{1}{2}$ coincides with [SSV 2013] which is quite non-trivial!

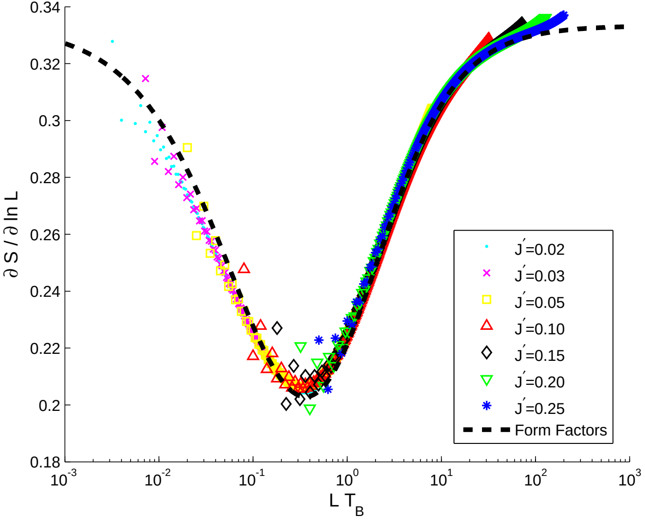
    "Bulk two-point function with defect scattering" versus "One-point function with boundary state"
  - Several wrong results in the literature, due to considerations of the type $S_A \sim c_{\text{eff}} \log L$ and failure to introduce scaling function $F(x)$

$$F(x) = \frac{2}{3}\int_0^\infty \mathrm{d}v\, \mathrm{e}^{-2v} \left(\frac{x}{v+x}\right)^2 + \frac{2}{3}\int_0^\infty \mathrm{d}v\, \left(\alpha \mathrm{e}^{-4\alpha v} + (1-\alpha)\mathrm{e}^{-4(1-\alpha)v}\right) \left(\frac{v}{v+x}\right)^2$$

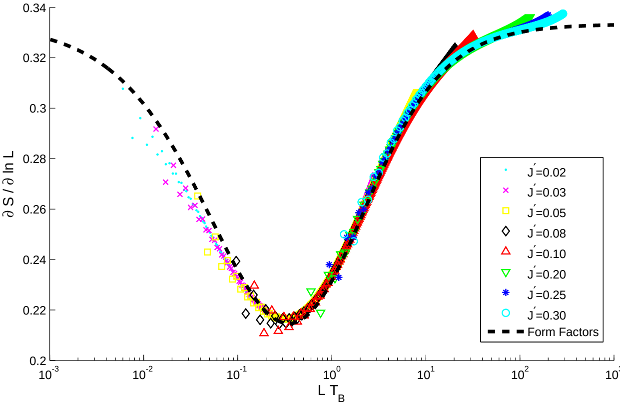


Note that $F(x)|_{\alpha\to 0} = \frac{1}{6} + F(x)|_{\alpha=0}$.

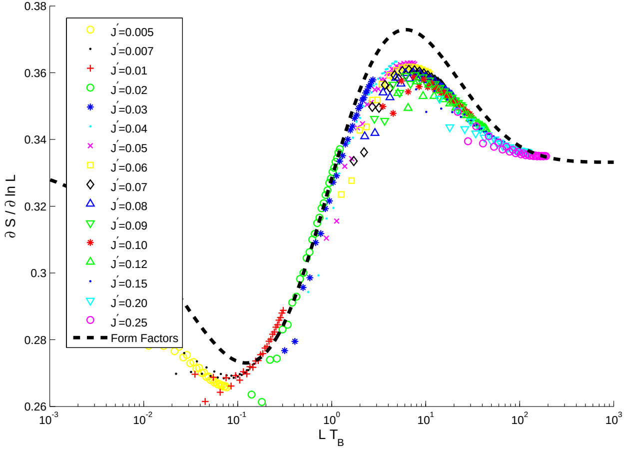
Two weak links: comparison with numerics ($\alpha = \frac{1}{2}$)



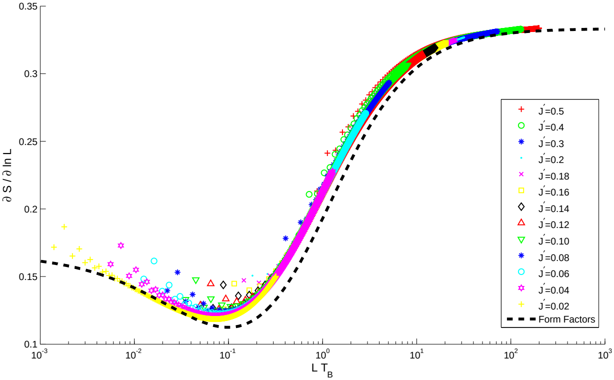
Two weak links: comparison with numerics ($\alpha = \frac{1}{4}$)



Two weak links: comparison with numerics ($\alpha = \frac{1}{100}$)



Two weak links: comparison with numerics ($\alpha = 0$)



UV and IR limits

- Consider e.g. $\alpha = 0$
- UV limit ($x \ll 1$):

$$F(x) = \frac{1}{6} + \frac{4}{3}\left(1 + \gamma + \log(4x)\right) x + \mathcal{O}(x^2)$$

- The logarithmic term shows that S_A is non perturbative
  - Already pointed out in [SSV 2013] for the case $\alpha = \frac{1}{2}$
- IR limit ($x \gg 1$):

$$F(x) = \frac{1}{3} - \frac{1}{3x} + \mathcal{O}\left(\frac{1}{x^2}\right)$$

General scaling argument

$S = -\frac{\mathrm{d}}{\mathrm{d}n} R_n\Big|_{n=1}$ obtained from $R_n = \mathrm{Tr}\ \rho^n$, where

$$R_n = c_n \left(\frac{L}{a}\right)^{-\frac{1}{6}(n-n^{-1})} \Omega(LT_B, n)$$

The non-universal c_n depends on aT_B. We find

$$S = h(LT_B) + k(aT_B)$$

with

$$\begin{aligned} h(LT_B) &= -\frac{\mathrm{d}}{\mathrm{d}n}\Omega\Big|_{n=1} + \frac{1}{3}\log(LT_B)\,, \\ k(aT_B) &= -\frac{\mathrm{d}}{\mathrm{d}n}c_n\Big|_{n=1} - \frac{1}{3}\log(aT_B)\,. \end{aligned}$$

Deriving with respect to $\log L$ we get a function of LT_B as claimed.

Mutual information

- Mutual information of subsystems A and B: $I \equiv S_A + S_B - S_{A\cup B}$
- $A = [-\ell_A, 0]$ and $B = [0, \ell_B]$
- I is upper bound on $S_{A,B}$ (negativity is better)
- Easy to express I in terms of $S_A(\ell, L)$ previously computed
- One weak link: $I = \frac{\cos^2 \xi}{3} \log \frac{\ell_A \ell_B}{\ell_A + \ell_B}$
  - Cf. negativity $\mathcal{E} \sim \frac{c}{4} \log \frac{\ell_A \ell_B}{\ell_A + \ell_B}$ [Calabrese-Cardy-Tonni 2012]
- Two weak links (e.g. with $\ell_A = \ell_B = L$):

$$\frac{\partial I}{\partial \ln L} := G(x) = \frac{4}{3} \int_0^\infty \mathrm{d}v \left(\mathrm{e}^{-2v} - \mathrm{e}^{-4v} \right) \left(\frac{x}{v+x} \right)^2$$

- We have $G(0) = 0$ and $G(\infty) = \frac{1}{3}$

Conclusion and outlook

- Used massless Form Factor approach to compute correlation function of branch-cut twist operators, in the presence of scattering on a quantum impurity.
- Derived the universal scaling function $F(x)$ of the entanglement entropy of a resonant level (quantum dot), with an asymmetrically placed sub-system.
- Validated $F(x)$ against large-scale numerics ($N = 64\,000$ sites)

Perspectives

- Fidelity (talk by H. Saleur)
- Local quench
- Understand bound states for $J' > J$
- Take FF computation to higher order
- Two disjoint intervals, negativity,. . .