Exact overlaps in the anisotropic Kondo problem

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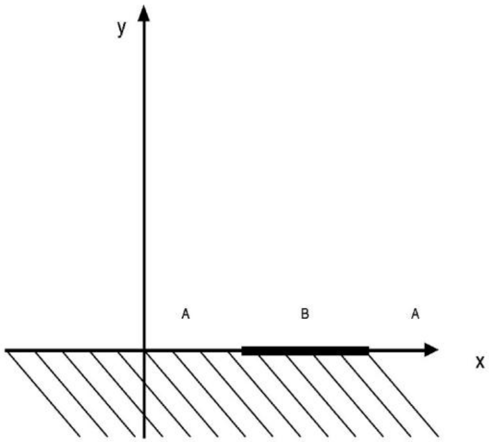
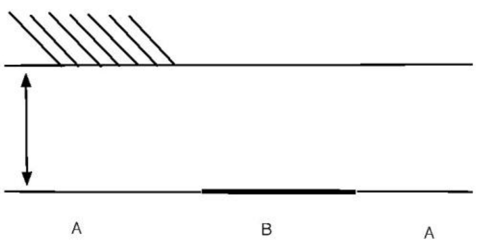
The set-up

- Conformal boundary conditions changing operators



$$\langle O_{AB}(x_1) O_{BA}(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2d_{AB}}}$$

$$w = \frac{iL}{\pi} \ln z$$



$$\langle O_{AB}(w_1, \bar{w}_1) O_{BA}(w_2, \bar{w}_2)\rangle \approx \left(\frac{\pi}{L}\right)^{2d_{AB}} \exp\left[-\frac{\pi}{L} d_{AB} |w_1 - w_2|\right]$$

$$|{}_{AA}\langle 0|0\rangle_{AB}| = \left(\frac{\pi}{L}\right)^{d_{AB}}$$

$$d_{AB} = \frac{L}{\pi}\left(E^0_{AB} - E^0_{AA}\right)$$

(Cardy, Affleck Ludwig)

- Forgetting about the left boundary condition (fixed in what follows): ground states with different conformal boundary conditions are orthogonal. This is the Anderson Orthogonality catastrophe (Anderson 67)

This has nothing to do with interactions. Can be understood simply for free fermions (Landau Fermi liquid) as a collective effect : cumulated shift of all the one electron states hidden in the Fermi sea.

⟨○ ○ ○ ○○ ○|○○○○○○○⟩ $\rightarrow$ $$|_{AA}\langle 0|0\rangle_{AB}| \approx \exp\left[-\frac{1}{2\pi^2}\int \frac{d\omega}{\omega}(\delta_A(\omega)-\delta_B(\omega))^2\right]$$

energy above Fermi surface   phase shifts

or $$|_{AA}\langle 0|0\rangle_{AB}| \propto L^{-(\delta_A^F-\delta_B^F)^2/2\pi^2}$$

- Anisotropic Kondo problem: there is an interaction on the boundary, and fermions do not see conformal boundary conditions:

$\delta = \frac{\pi}{2}$ $\longrightarrow$ $\delta = 0$

ω

at the free point, problem admits a full Fermi liquid description:

$$e^{2i\delta} = i \tanh\left(\frac{\theta - \theta_K}{2} - \frac{i\pi}{4}\right) \qquad \omega = \mu e^{\theta} \qquad T_K = \mu e^{\theta_K}$$ (Kondo temperature)

If one of the Kondo temperatures is zero (no Kondo coupling), shifts at the Fermi surface differ by $\frac{\pi}{2}$

$$|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}=0}| \propto L^{-\frac{1}{8}}, \quad \text{free case}$$

If one of the Kondo temperatures is much greater than the other:

orthogonality exponent

$$|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}| \propto \left(\frac{T_K^{(2)}}{T_K^{(1)}}\right)^{-\frac{1}{8}}, \; T_K^{(2)} >> T_K^{(1)} \quad \text{free case}$$

- In general, and in the scaling limit such a scalar product is a universal function of the ratio $T_K^{(2)}/T_K^{(1)}$ (it is not perturbative in either of these temperatures)

what is this function in the free case, in the interacting case?

- Anisotropic Kondo: 3D spinful Fermi liquid interacting with localized magnetic impurity. Spherical waves + reduction to s mode + bosonization + decoupling of the charge degrees of freedom + SU(2) interaction broken down to U(1) leaves

$$H = \frac{1}{2}\int_{-\infty}^{0} \left[(\partial_x \Phi)^2 + \Pi^2\right] + J_\perp \left(e^{i\sqrt{2\pi}\Phi(0)}\sigma^- + e^{-i\sqrt{2\pi}\Phi(0)}\sigma^+\right) + \frac{J_z}{2}\Pi(0)\sigma^z$$

Unfolding + canonical transformation

chiral boson

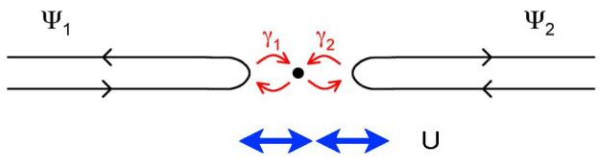
$$U_{J_z}^\dagger H U_{J_z} = \int dx (\partial_x \phi)^2 + J_\perp \left(e^{i\beta\phi(0)}\sigma_- + e^{-i\beta\phi(0)}\sigma_+\right)$$

Dimension of the perturbation is

$$\frac{\beta^2}{8\pi} = \left(1 - \frac{J_z}{\sqrt{2\pi}}\right)^2$$

Crossover (Kondo) temperature: $T_K \propto J_\perp^{1/1-\frac{\beta^2}{8\pi}}$

- This hamiltonian occurs in a variety of other contexts: two state problem in dissipative quantum mechanics, IRLM,...



- The scalar product is a particular case of matrix elements $_{T_K^{(2)}}\langle\Psi^{(2)}|\Psi^{(1)}\rangle_{T_K^{(1)}}$

eigenstates

In the free case: $_{T_K^{(2)}}^{\epsilon_1,\ldots,\epsilon_n}\langle\theta_1,\ldots,\theta_n|\theta'_1,\ldots,\theta'_m\rangle_{T_K^{(1)}}^{\epsilon'_1,\ldots,\epsilon'_m}$

In fact, massless particles description valid for general anisotropic Kondo which is integrable

These matrix elements are crucial in the study of quenches: example of the Kondo exciton (absorption of a photon ~ turning on Kondo coupling)

- The ratios $\dfrac{_{T_K^{(2)}}^{\epsilon_1,\ldots,\epsilon_n}\langle\theta_1,\ldots,\theta_n|\theta'_1,\ldots,\theta'_m\rangle_{T_K^{(1)}}^{\epsilon'_1,\ldots,\epsilon'_m}}{_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}}$

can (in principle) be determined by an axiomatic form-factors approach (Lesage Saleur 98)

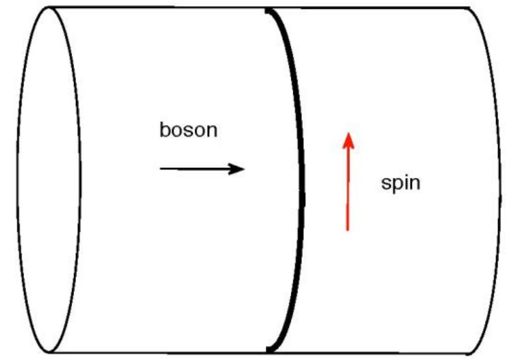
This was used to calculate the Loschmidt echo and the work distribution in the Kondo exciton problem (Vasseur, Trinh, Haas, Saleur 13)

Some information on $|_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}|$ can then be obtained by resumming the series - not too efficient however.

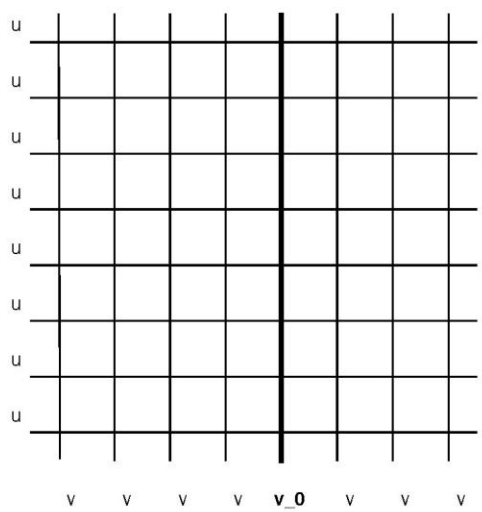
- Can one get the overlaps directly and exactly?

A little formalism

- In imaginary time, the insertion of the impurity can be thought of in terms of a monodromy matrix M. It acts on the spin degrees of freedom, and its elements are operators acting in the (right moving) free boson Hilbert space. (Bazhanov Lukyanov Zamolochikov 94)



- This is exactly the continuum limit of the six vertex model monodromy matrix, in the particular case of a vertical line carrying a large bare rapidity



The monodromy matrix can be expressed as

$$M(J_\perp) = e^{2i\pi P\sigma^z}\mathcal{P}\exp\left[J_\perp\int_0^{2\pi/T}\left(q^{\sigma^z/2}e^{i\beta\phi(0)}\sigma_- + q^{-\sigma^z/2}e^{-i\beta\phi(0)}\sigma_+\right)\right] \qquad q = -e^{-8i\beta^2}$$

Integrability of Kondo arises in this context from the zero curvature representation of SG

$$(\partial_x^2 - \partial_t^2)\Phi + \frac{m^2}{\beta}\sin(\beta\Phi) = 0$$

$$[\partial_+ - A_+, \partial_- - A_-] = 0$$

with

$$A_+ = \frac{1}{4i}\left[\beta(\partial_x\Phi + \partial_t\Phi)\sigma^3 - me^{\theta}(e^{-i\beta\Phi/2}\sigma^- + e^{i\beta\Phi/2}\sigma^+)\right]$$
$$A_- = \frac{1}{4i}\left[\beta(\partial_x\Phi - \partial_t\Phi)\sigma^3 + me^{-\theta}(e^{i\beta\Phi/2}\sigma^- + e^{-i\beta\Phi/2}\sigma^+)\right]$$

non chiral field

spectral parameter $me^{\theta} \propto J_\perp$

- The geometry we are interested in is

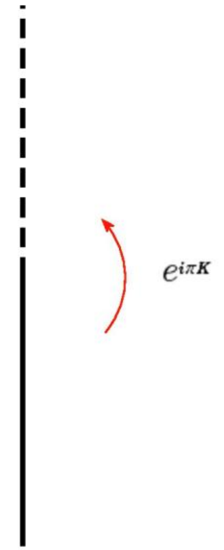
requires splitting the monodromy matrix into two objects propagating (in the vertical sense) from $y=-\infty$ to $y=0$ and from $y=0$ to $y=\infty$

In the classical case, one writes $M(\theta)=T(\theta)Q(\theta)$

Jost solutions

so we need a quantum version of the Jost functions (Lukyanov, Shatashvili 93,94)

Note: in order to have T and Q act on the same space one needs to turn to radial quantization (corner transfer matrix)



General arguments lead to $T(\theta) = \begin{pmatrix} iT_+(\theta) & iT_-(\theta) \\ T_+(\theta+i\pi) & -T_-(\theta+i\pi) \end{pmatrix}$

$$C^{ab}T_a(\theta+i\pi)T_b(\theta) = i, \quad C^{ab} = \delta_{a+b}$$

where now $me^{\theta} \propto J_{\perp}^{1/1-\frac{\beta^2}{2\pi}}$

Relations satisfied by T

- Recall that anisotropic Kondo can be studied using massless scattering (massless limit of the soliton/antisoliton description of SG) (Faddeed Takhtajan, Andrei, Fendley, Fendley Saleur, Zamo^2...) For R moving particles obeying $e = p = me^{\theta}$

$$Z_a^\dagger(\theta_1) Z_b^\dagger(\theta_2) = S_{ab}^{cd}(\theta_1 - \theta_2) Z_d^\dagger(\theta_2) Z_c^\dagger(\theta_1)$$

$$\begin{aligned} S_{++}^{++}(\theta) &= S(\theta) \\ S_{+-}^{+-}(\theta) &= S(\theta) \frac{\sinh \frac{\theta}{\xi}}{\sinh \frac{i\pi - \theta}{\xi}} \\ S_{+-}^{-+}(\theta) &= S(\theta) \frac{\sinh \frac{i\pi}{\xi}}{\sinh \frac{i\pi - \theta}{\xi}} \end{aligned}$$

$$\frac{\beta^2}{8\pi} \equiv \frac{\xi}{\xi + 1}$$

$$S(\theta, \xi) = -\exp\left(-i \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \sin \omega\theta \frac{\sinh[(\pi - \pi/\xi)\omega/2]}{2\sinh(\pi\omega/2\xi)\cosh(\pi\omega/2)}\right)$$

S matrix has quantum su(2) symmetry with $q_Z = e^{\frac{i\pi}{\xi}}$

- Massless kinks scatter on the Kondo impurity with (Andrei, Fendley)

Note: it does not depend on the anisotropy!

$$R(\theta) = i\tanh\left(\frac{\theta-\theta_0}{2} - \frac{i\pi}{4}\right) \qquad T_K = m e^{\theta_0}$$

- Conjectured relations for ZF and Joost operators:

impurity scattering

$$\begin{aligned} Z_a^\dagger(\theta_1)T_b(\theta_2) &= ab\ i\ \tanh\left(\frac{\theta_1-\theta_2}{2} - \frac{i\pi}{4}\right) T_b(\theta_2)Z_a^\dagger(\theta_1) \\ T_a(\theta_1)T_b(\theta_2) &= R_{ab}^{cd}(\theta_1-\theta_2)T_d(\theta_2)T_c(\theta_1) \end{aligned}$$

The lattice scattering matrix

with

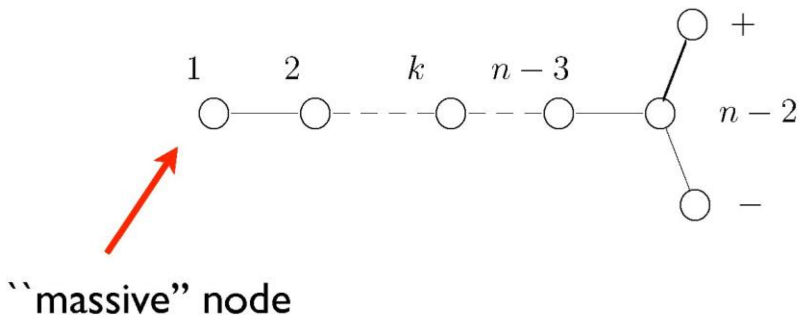
$$\begin{aligned} R_{++}^{++}(\theta) &= R(\theta) \\ R_{+-}^{+-}(\theta) &= -R(\theta)\frac{\sinh\frac{\theta}{\xi+1}}{\sinh\frac{i\pi-\theta}{\xi+1}} \\ R_{+-}^{-+}(\alpha) &= R(\theta)\frac{\sinh\frac{i\pi}{\xi+1}}{\sinh\frac{i\pi-\theta}{\xi+1}} \end{aligned}$$

quantum su(2) symmetry with $q_M = e^{\frac{i\pi}{\xi+1}} = q$

$$R(\theta) = -S(-\theta, \xi+1)$$

The ZF and Joost operators have different quantum group symmetries $q_Z = e^{\frac{i\pi}{\xi}}$ $q_M = e^{\frac{i\pi}{\xi+1}} = q$

like the SG S matrices and the 6 vertex R matrices. Can be seen from the TBA ($\xi = n-1$)



The main result

- Bosonization of form factors (Lukyanov) leads to ($T_K^{(2)}/T_K^{(1)} = e^{\theta_{21}}$)

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = \langle T_\pm(\theta_1)T_\mp(\theta_2+i\pi)\rangle = (1+\xi)\,\frac{\sinh\frac{\theta_{12}}{2(1+\xi)}}{\sinh\frac{\theta_{12}}{2}}\,G(\theta_{12})$$

with minimal solution $$G(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\frac{\sin^2(\theta t/\pi)}{\sinh 2t\cosh t}\frac{\sinh t\xi}{\sinh t(\xi+1)}\right]$$

symmetry $\theta_{12}\to\theta_{21}$

- Checks

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| \propto \exp\left[-\frac{\xi}{4(\xi+1)}\theta_{21}\right],\quad \theta_{21}\to\infty$$ so

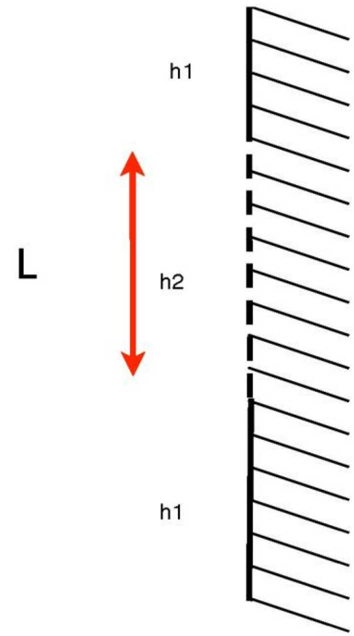
$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| \propto \left(\frac{T_K^{(2)}}{T_K^{(1)}}\right)^{-\frac{\beta^2}{32\pi}},\quad T_K^{(2)} >> T_K^{(1)}$$ and $$h_{Kondo} = \frac{1}{4}\frac{\beta^2}{8\pi}$$

dimension of the boundary condition changing operator from weak to strong coupling Kondo fixed point

- Perturbative calculations require both Kondo couplings to be non zero to avoid the catastrophe.

The expansion variable is then $\frac{J_\perp^{(2)}-J_\perp^{(1)}}{J_\perp^{(1)}}$ requiring knowledge of correlation functions for non zero Kondo coupling to start with!

Can be done in the free fermion case where the calculation can be reformulated in terms of an Ising model with two different boundary fields. The scalar product is essentially the term of order one in L for the partition function



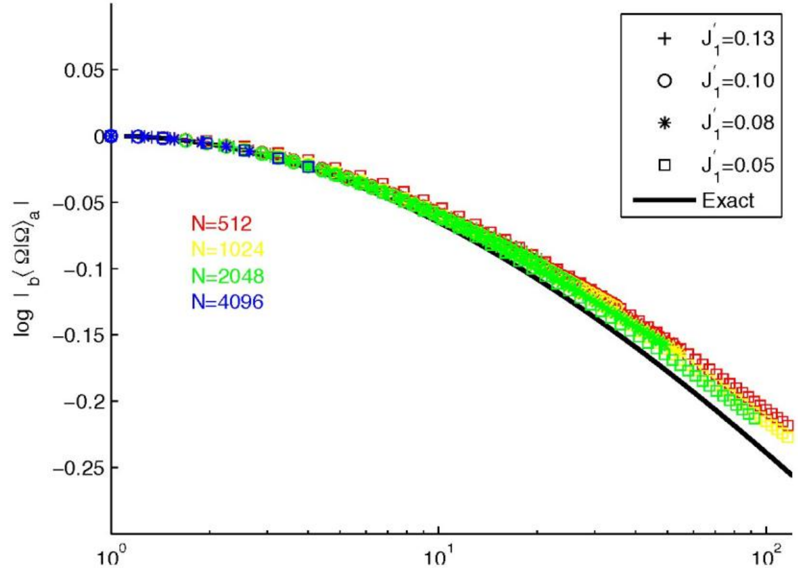
$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = 1 - \frac{\theta_{21}^2}{8\pi^2}, \quad \theta_{12} << 1$$

Can also be done in the semiclassical case

$$\left|{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}\right| = 1 + \frac{\xi}{2} - \frac{\xi}{4}\theta_{12}\coth\frac{\theta_{12}}{2} + (\xi^2) \qquad \xi \approx \frac{\beta^2}{8\pi}$$

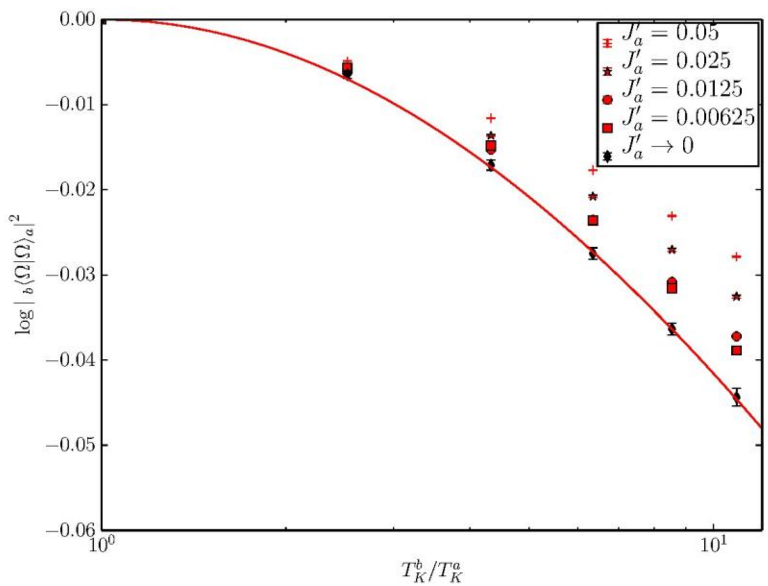
Numerics: difficult because scalar product evolves slowly, and finite size effects are very big (bare coupling must be very small, but Kondo length much smaller than system size!)

free case:



$h = 1/4$

Double extrapolation required.
Length up to 800

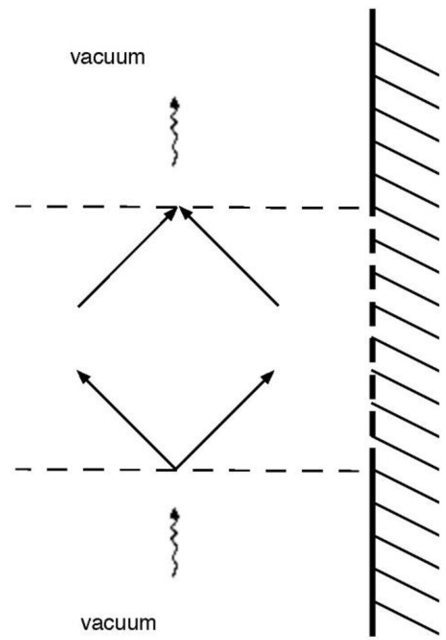


Form factors for BC changing operators

- Follow by ordinary axiomatic approach

Eg leading diagram for Loschmidt echo:

$$\frac{{}_{T_K^{(2)}}^{\epsilon_1,\ldots,\epsilon_n}\langle\theta_1,\ldots,\theta_n|\theta'_1,\ldots,\theta'_m\rangle_{T_K^{(1)}}^{\epsilon'_1,\ldots,\epsilon'_m}}{{}_{T_K^{(2)}}\langle vac|vac\rangle_{T_K^{(1)}}}$$



leading to work distribution etc.

- Note that the ratios are well defined in the conformal limit, even if scalar products all vanish. Example for Ising model

$$\frac{{}_+\langle\theta_1,\theta_2|vac\rangle_f}{{}_+\langle vac|vac\rangle_f} = i\ \tanh\frac{\theta_{12}}{2} \quad = \frac{0}{0}$$

free/fixed BC

so a naive check of unitarity says leads to

$${}_f\langle vac|vac\rangle_f = |{}_+\langle vac|vac\rangle_f|^2 \left(1 + \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} \tanh^2\frac{\theta_{12}}{2} + \ldots\right)$$

0 infinity

- In general the approach remains plagued by IR divergences: Anderson catastrophe strikes back!

For instance the Loschmidt echo (in imaginary time) for a quench in the free fermion case (Ising) will involve

$$ {}_f\langle vac|e^{-H_{T_b}\tau}|vac\rangle_f = \sum_{n=0}^{\infty}\int \frac{d\theta_i}{2\pi} e^{-\tau m \sum e^{\theta_i}} |_{T_b}\langle \theta_1, \ldots, \theta_n|vac\rangle_f|^2 $$

can be calculated by writing it as

$$ \frac{\sum_{n=0}^{\infty}\int \frac{d\theta_i}{2\pi} e^{-\tau m \sum e^{\theta_i}} \frac{|_{T_b}\langle \theta_1, \ldots, \theta_n|vac\rangle_f|^2}{|_{T_b}\langle vac|vac\rangle_f|^2}}{\text{same with } \tau = 0} $$

known from generalized FF axioms

IR divergences can be subtracted
by simultaneous expansion of numerator
and denominator

leading to (Vasseur et al. 2013)

$$\int_0^\infty \frac{du}{2\pi u}(e^{-xu}-1)\Psi(u) + \frac{1}{2!}\int_0^\infty \frac{du_1}{2\pi u_1}\int_0^\infty \frac{du_2}{2\pi u_2}(e^{-x(u_1+u_2)}-1)\left(\left(\frac{u_1-u_2}{u_1+u_2}\right)^2 - 1\right)\Psi(u_1)\Psi(u_2)$$

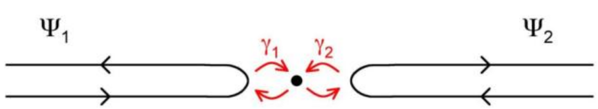
$$+\frac{1}{3!}\int_0^\infty \frac{du_1}{2\pi u_1}\int_0^\infty \frac{du_2}{2\pi u_2}\int_0^\infty \frac{du_3}{2\pi u_3}(e^{-x(u_1+u_2+u_3)}-1)\left[\left(\frac{u_1-u_2}{u_1+u_2}\right)^2\left(\frac{u_1-u_3}{u_1+u_3}\right)^2\left(\frac{u_2-u_3}{u_2+u_3}\right)^2 + 2 - \left(\frac{u_1-u_2}{u_1+u_2}\right)^2 - \left(\frac{u_1-u_3}{u_1+u_3}\right)^2 - \left(\frac{u_2-u_3}{u_2+u_3}\right)^2\right]\Psi(u_1)\Psi(u_2)\Psi(u_3) + \dots$$

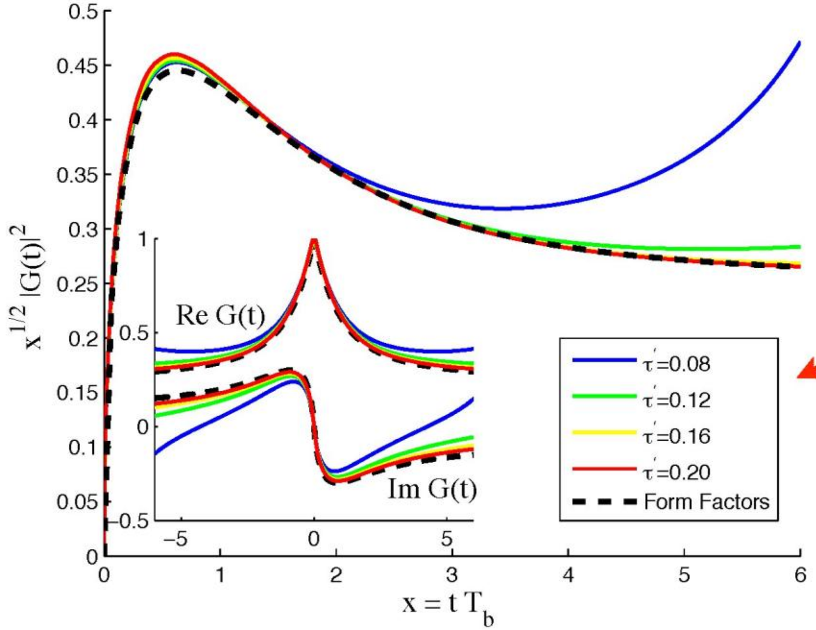
$$x = \tau T_b$$

where

$$\Psi(u) = \frac{\sqrt{u}}{1+u^2}\exp\left[\int_{-\infty}^\infty \frac{dt}{2t}\left(\frac{2}{t} - \frac{\cos\frac{\ln u}{2\pi}t}{\cosh\frac{t}{4}\sinh\frac{t}{2}}\right)\right],$$

- This converges also in real time, giving access eg to the Loschmidt echo for a sudden quench in the RLM





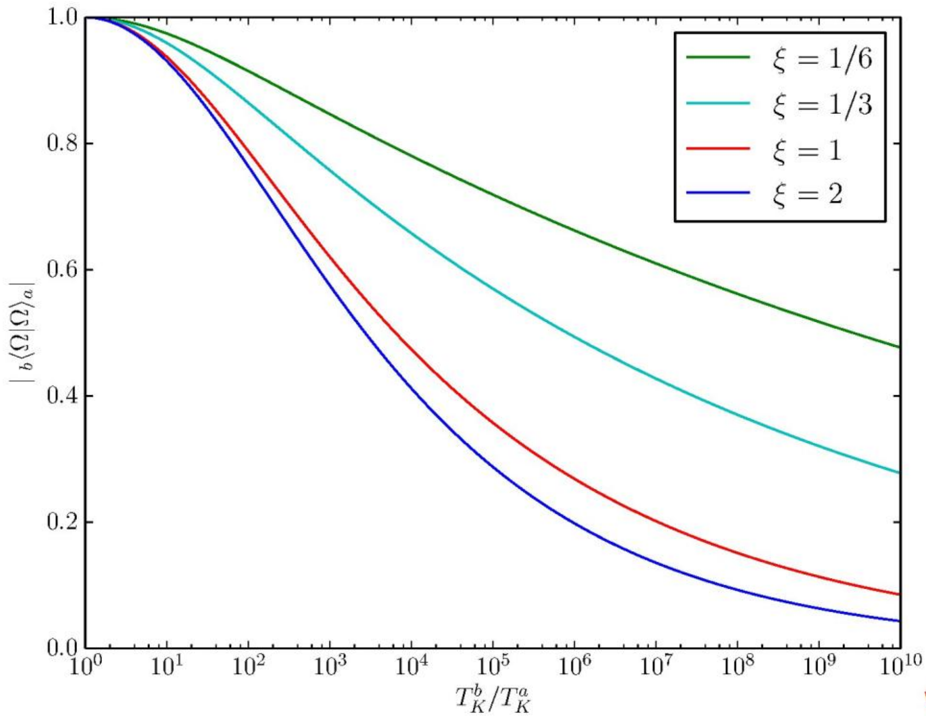
quenched tunneling amplitude, which is the same as the bare Kondo coupling

$|{}_f\langle vac|e^{-iH_{T_b}t}|vac\rangle_f| \propto t^{-1/4}$ at large times follows from CFT (Anderson exponent again)

the work distribution then has a bump around the Kondo temperature (Tureci et al. 2011)

Conclusion

- Not sure what this overlaps measures in Kondo from the point of view of entanglement...



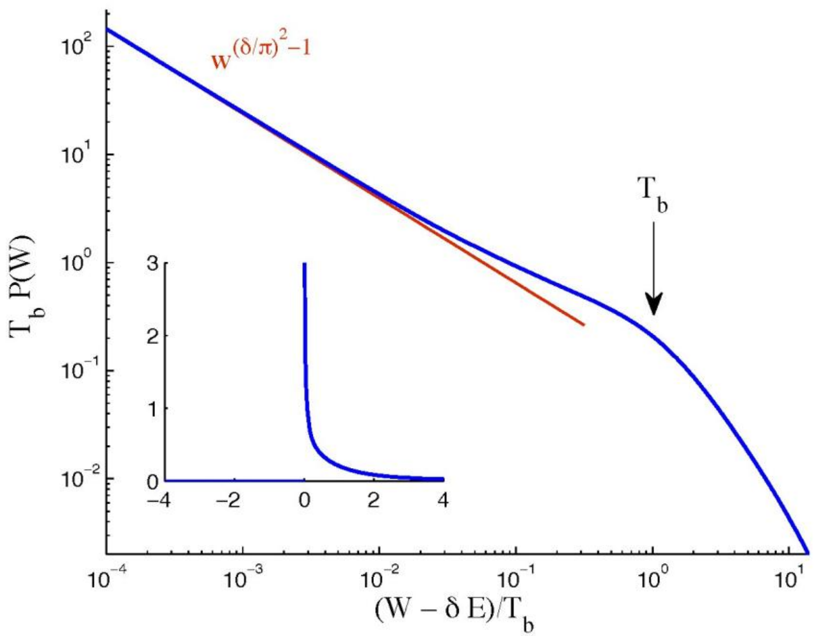
scale!

- In the two state problem of dissipative QM

coupling to harmonic bath

$$U_\beta H U_\beta^\dagger \equiv H_K = \int dx (\partial_x \phi)^2 + J_\perp \sigma_1 + \frac{\beta}{2} \partial_x \phi(0) \sigma_z$$

the decrease of the overlap as β increases expresses the loss of coherence of the two state system



$w^{(\delta/\pi)^2-1}$
T_b
$T_b\,P(W)$
10^2
10^1
10^0
10^{-1}
10^{-2}
3
2
1
0
−4
−2
0
2
4
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
$(W - \delta E)/T_b$