Primes go Quantum:

there is entanglement in the primes

Germán Sierra
In collaboration with José Ignacio Latorre
Madrid-Barcelona-Singapore

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Plan

- A primer of prime numbers
- Primes and Physics
- Prime state
- Entanglement of the Prime state
- Conclusions

Based on arXiv: 1302.6245 and arXiv:1403.4765

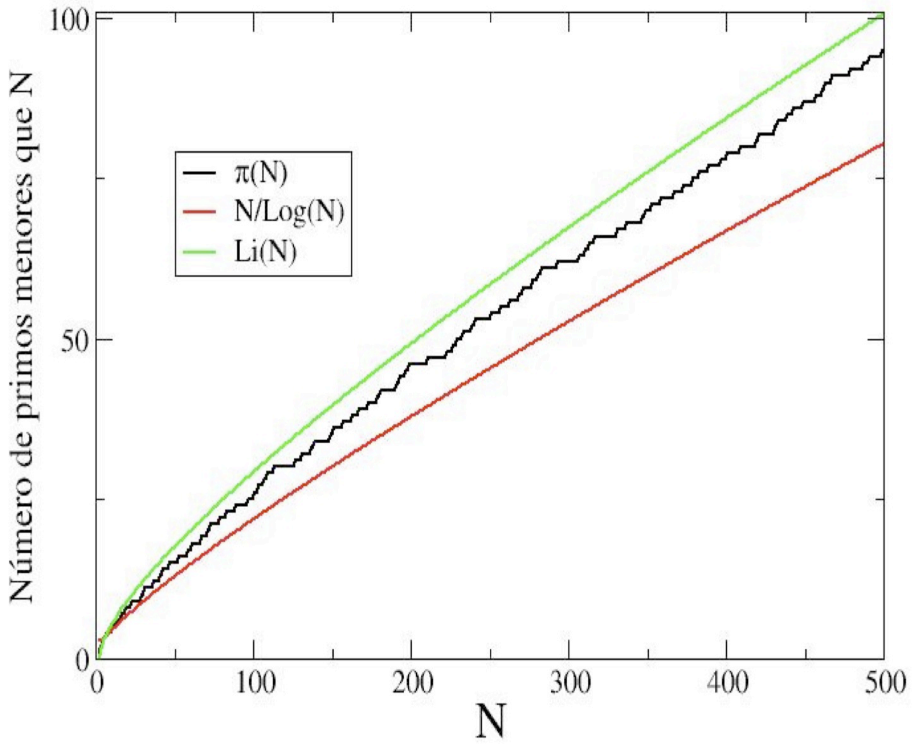
Prime counting function

$\pi(x)$: number of primes p less than or equal to x

$$\pi(100) = 25$$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Asymptotic behaviour: Gauss law



$$\pi(x) \approx Li(x) \approx \frac{x}{\ln x}$$

$$x \to \infty$$

Average behaviour

Prime Number Theorem (PNT)

- Hadamard (1896)
- de la Vallee-Poussin

Density of primes: $$\frac{d\pi(x)}{dx} \approx \frac{1}{\ln x}$$

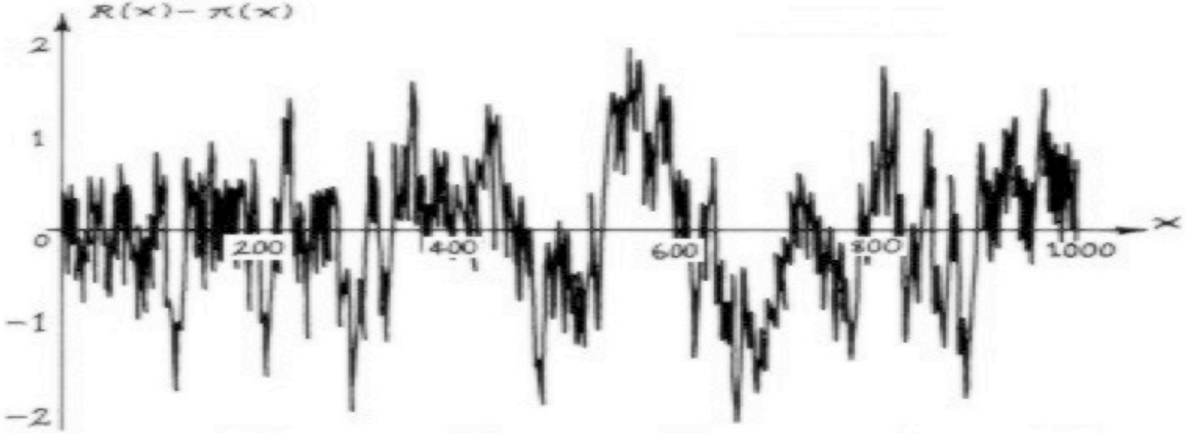
Largest known value $\pi(10^{24}) = 18\,435\,599\,767\,349\,200\,867\,886 \approx 1.8\;10^{22}$ Platt (2012)

$$Li(10^{24}) - \pi(10^{24}) \approx 1.7\;10^{10}$$

The prime number function must oscillate around the Li(x) infinitely many times (Littlewood)

A first change of sign is expected for occur below the Skewes number

The fluctuations of around are expect to be bounded by



This statement is equivalent to the Riemann hypothesis (RH)

The zeta function and the Riemann hypothesis

Rosetta stone for Maths

$$\varsigma(s) = \sum_{1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}\, s > 1$$

$$\varsigma(s) = \prod_{p=2,3,5,\ldots} \frac{1}{1-p^{-s}}, \mathrm{Re}\, s > 1$$

$$\varsigma(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+s/2)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

n: integers

p: primes

:Riemann zeros

Riemann hypothesis (1859):

the complex zeros of the classical zeta

function $\varsigma(s)$ all have real part equal to 1/2

$$\varsigma(s_n) = 0, s_n \in C \rightarrow s_n = \frac{1}{2} + i E_n, \quad E_n \in \Re, n \in Z$$

In fact:

A gas of primes (Julia, Spector, 1990)

Single particle levels

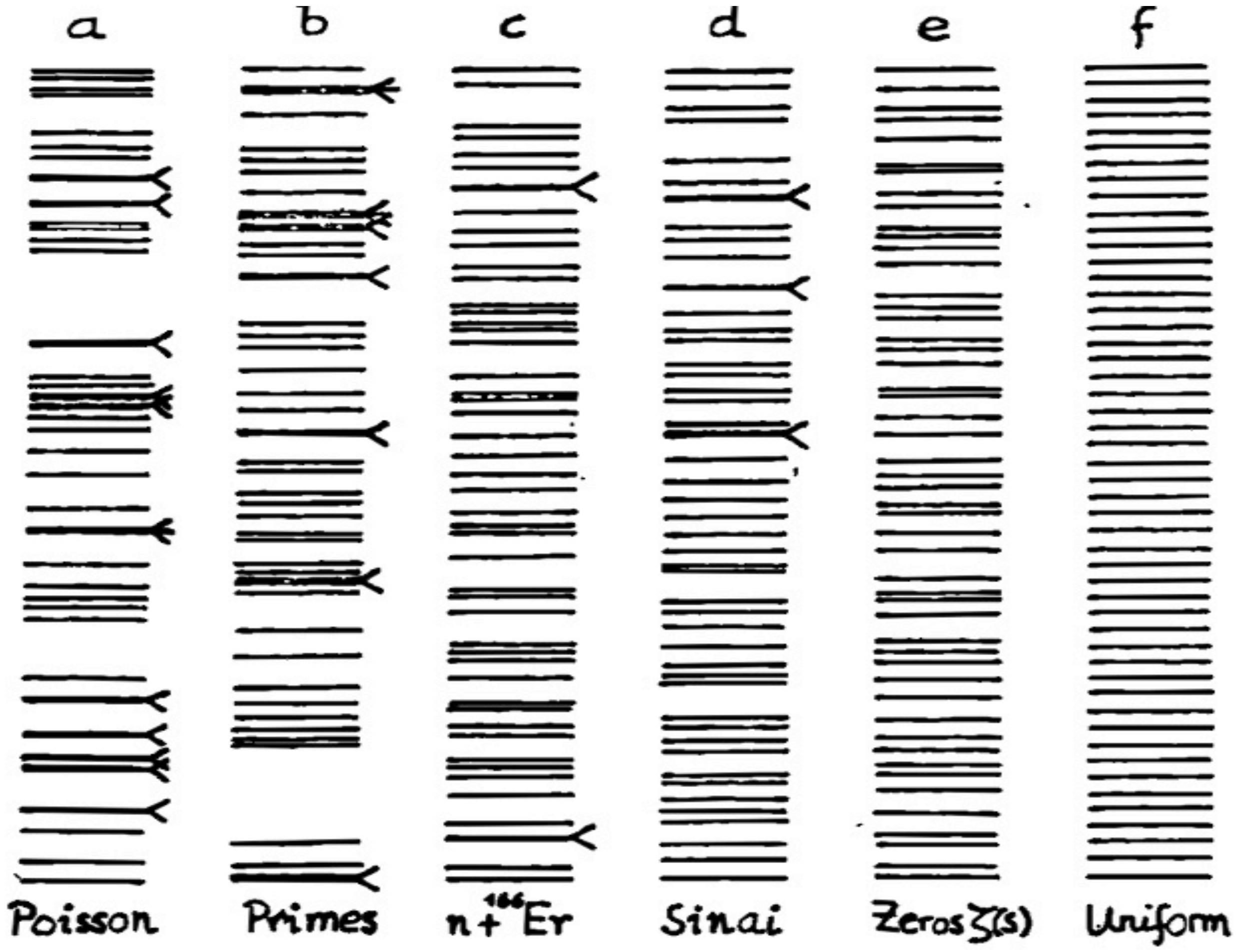
Many particle state of bosons

Total energy

Partition function

Z diverges at s=1 → Hagedorn transition

Bohigas, Gianonni (1984)



GUE statistics

The Riemann zeros look like the spectrum of random physical systems

Quantum chaos, primes and Riemann zeros

Berry Conjecture (1986):

there exist a classical chaotic Hamiltonian H

Periodic orbits → prime numbers → periods = log p

Energies → Riemann zeros → s = ½ + I E

Proof of the Riemann hypothesis

In previous models the primes are classical objects:

- Energies of Stat Mech (primon gas)

- Periods of orbits (quantum chaos)

Idea: make the primes quantum objects

Quantum Computation and prime numbers
(JIL, GS, 2013)

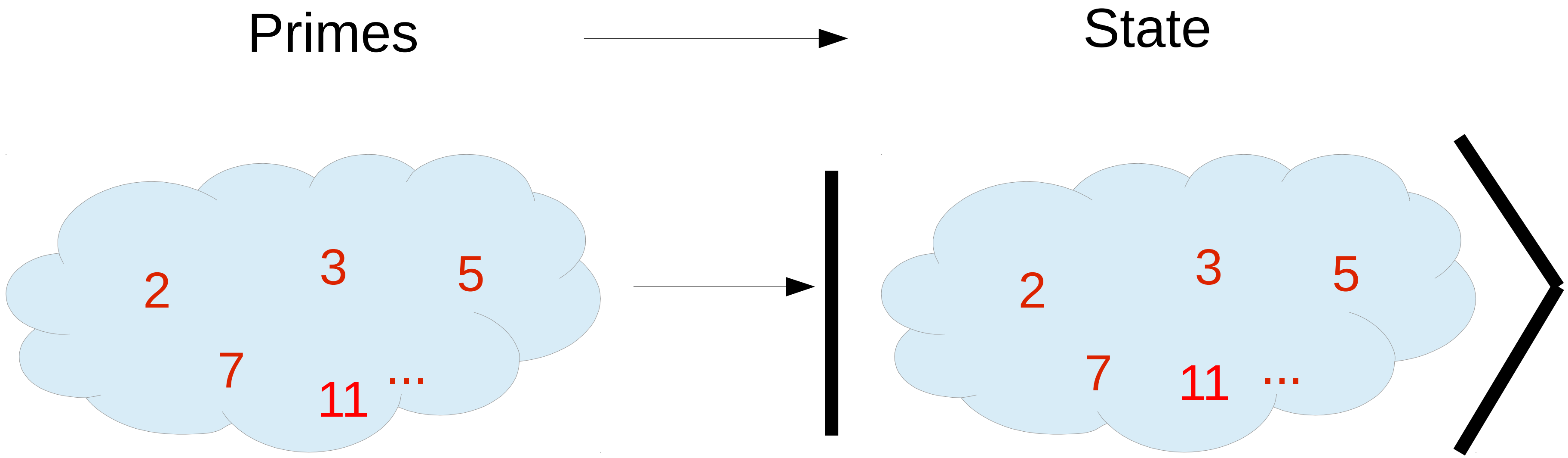
Classical computer

n bits $x = x_0 2^0 + x_1 2^1 + \ldots x_{n-1} 2^{n-1}, \quad x_i = 0,1, \quad x = 0,1,\ldots 2^n - 1$

Quantum computer

n qubits $|x\rangle = |x_{n-1},\ldots,x_0\rangle = |x_{n-1}\rangle \otimes \ldots \otimes |x_0\rangle$

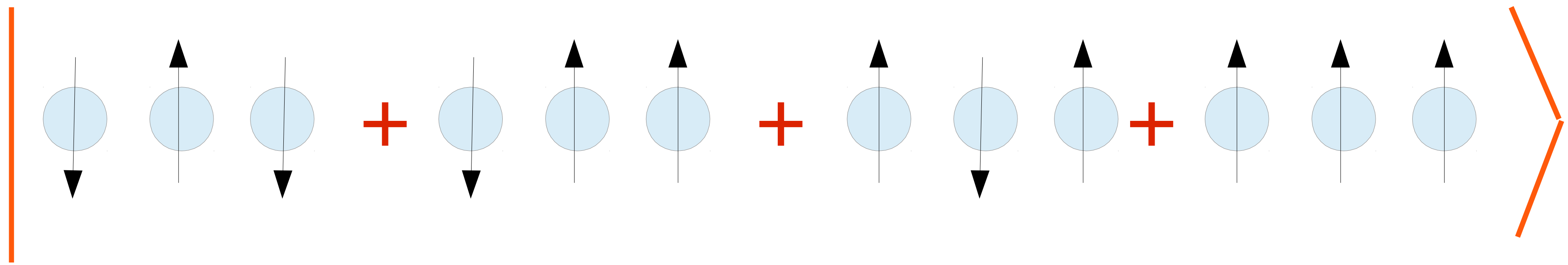
The Prime State



is the prime counting function

Quantum Mechanics allows for the superposition of primes implemented as states of a computational basis

Ex. n=3



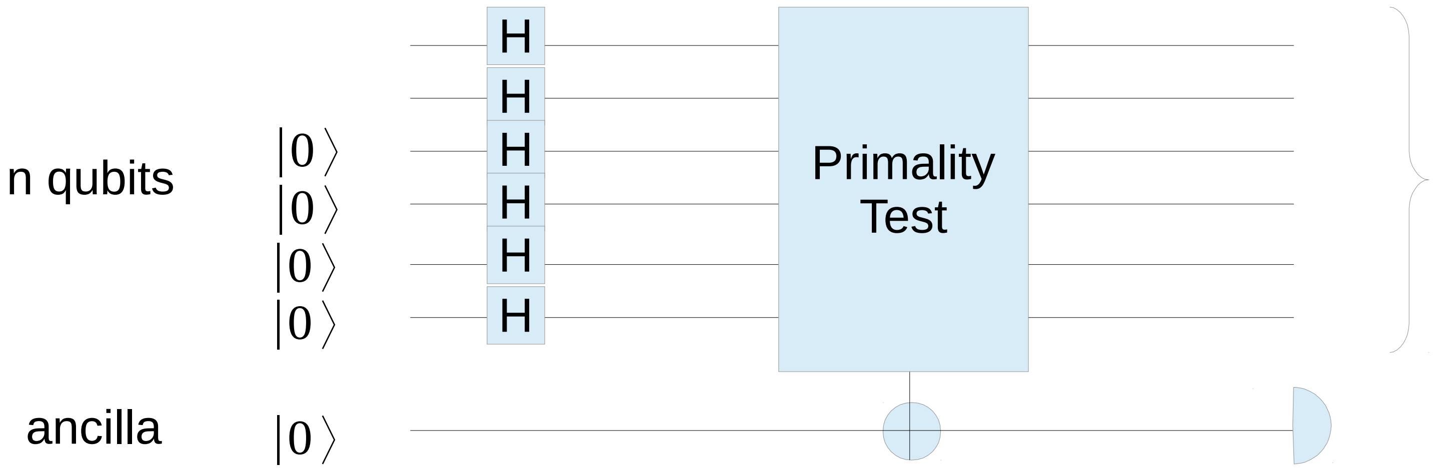
Could the Prime state be constructed?

Does it encode properties of prime numbers?

Could it provide the means to explore Arithmetics?

What are its entanglement properties?

Construction of the Prime state (probabilistic)



Efficient construction

Construction of the Prime state (deterministic)

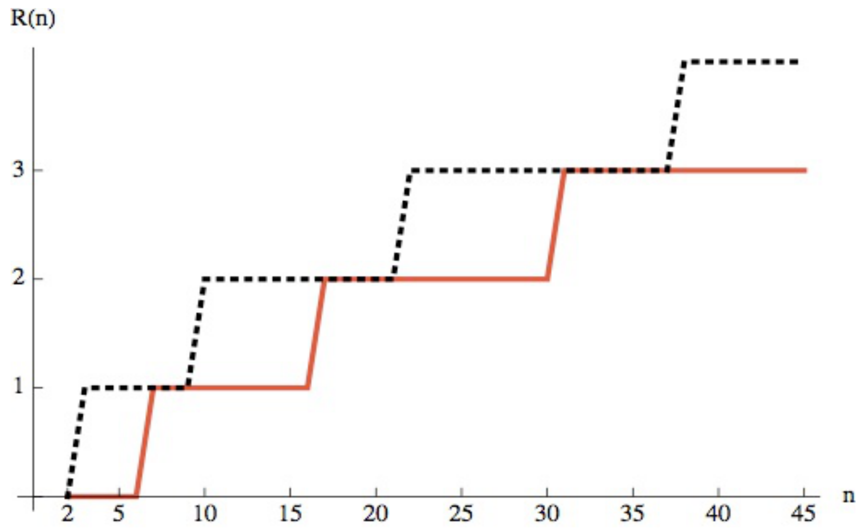
Grover's search algorithm

Oracle $U_{oracle}|x\rangle=(-1)^{\chi(x)}|x\rangle$

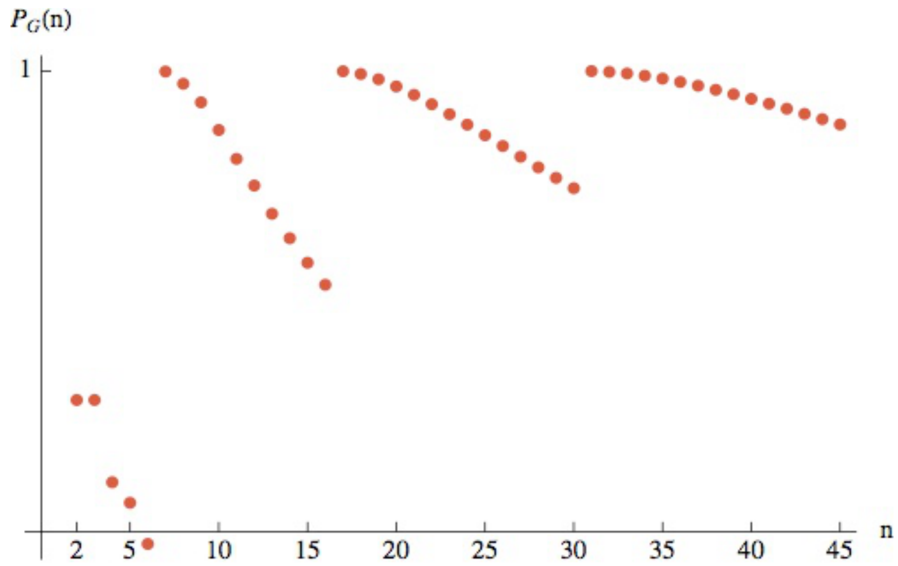
$$\chi(x)=1(prime),0(composite)$$

calls to the oracle

calls to Grover



Overlap between
Grover state and the Prime state



We need to construct an oracle!

Construction of a Quantum Primality oracle

Idea: take a classical primality test and make it quantum-> unitary transformations

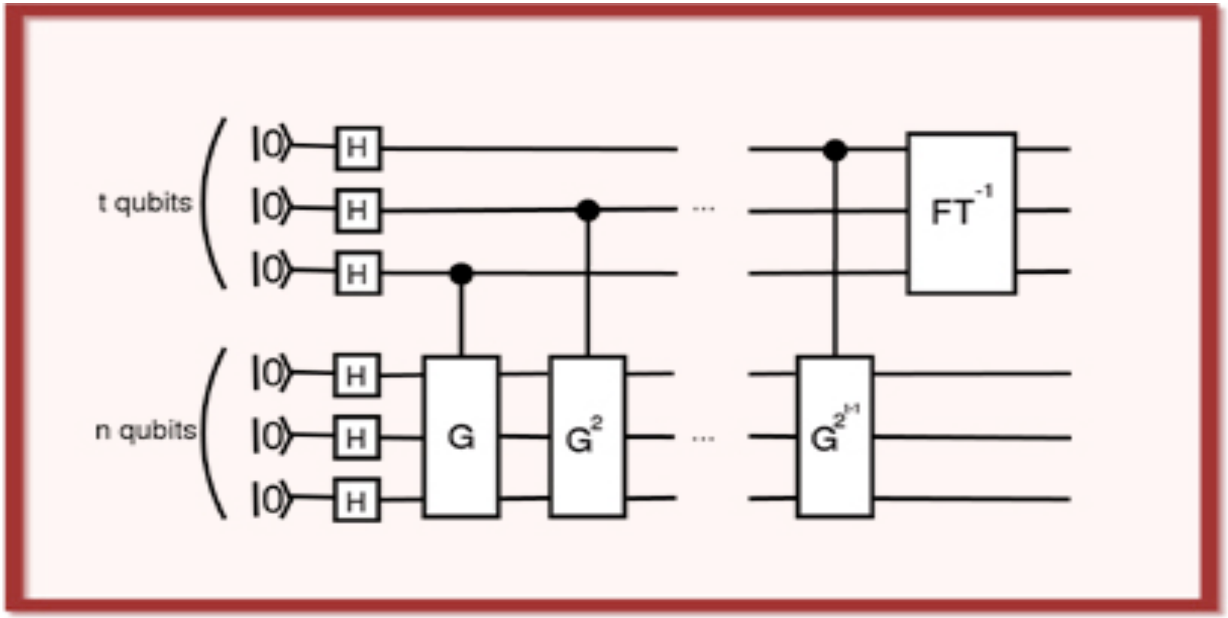
Miller-Rabin primality test

- Find s and d (odd) such that
- Choose witness a
- If $a^d \neq 1 \, mod \, x$ then x is composite with certainty
  $a^{2^r d} \neq -1 \, mod \, x \quad 0 \leq r \leq s-1$
- If the test fails, x may be prime or composite.
- Latter case: a is a strong liar to x
- Eliminate strong liars checking less than witnesses

Quantum Counting of Prime numbers

quantum primality oracle + quantum counting algorithm

Brassard, Hoyer, Tapp (1998)



Counts the number of solutions to the oracle

We want to count *M* solutions out of *N* possible states

We know an estimate $\tilde{M}$

$$|\tilde{M} - M| < \frac{2\pi}{c} M^{\frac{1}{2}} + \frac{\pi^2}{c^2}$$

Bounded error in quantum counting

Bounded error in the quantum counting of primes

$$\left| \pi_{QM}(x) - \pi(x) \right| \leq \frac{2\pi}{c} \frac{x^{1/2}}{\log^{1/2} x}$$

We use the PNT

$$\left|\pi_{QM}(x) - \pi(x)\right| \leq \frac{2\pi}{c} \frac{x^{1/2}}{\log^{1/2} x}$$

Riemann Hypothesis

$$|Li(x) - \pi(x)| < \frac{1}{8\pi} \sqrt{x} \log x$$

Error of quantum counting < fluctuations under the RH

A quantum computer could falsify the RH, but not prove it !!

Classical versus quantum computation of $\pi(x)$

Best classical algorithm by Lagarias-Miller-Odlyzko (1987)
implemented by Platt (2012)

time $T \sim x^{\frac{1}{2}}$ space $S \sim \log x$

A Quantum Computer could calculate the size of fluctuations more efficiently than a classical computer

Construction of the Twin Prime state

Twin Prime counting function
= p, p+2 < x primes

$$\pi_2(x) \approx 2C_2 \frac{x}{(\log x)^2}$$

$$\Pr(|twin\ primes\rangle) = \frac{\pi_2(2^n)}{\pi(2^n)} \approx \frac{2C_2}{n \log 2}$$

Entanglement of a single qubit:

Density matrix qubit i=1

Odd primes

$$\pi_{4,1} : 5, 13, 17, \ldots \quad 4n+1$$

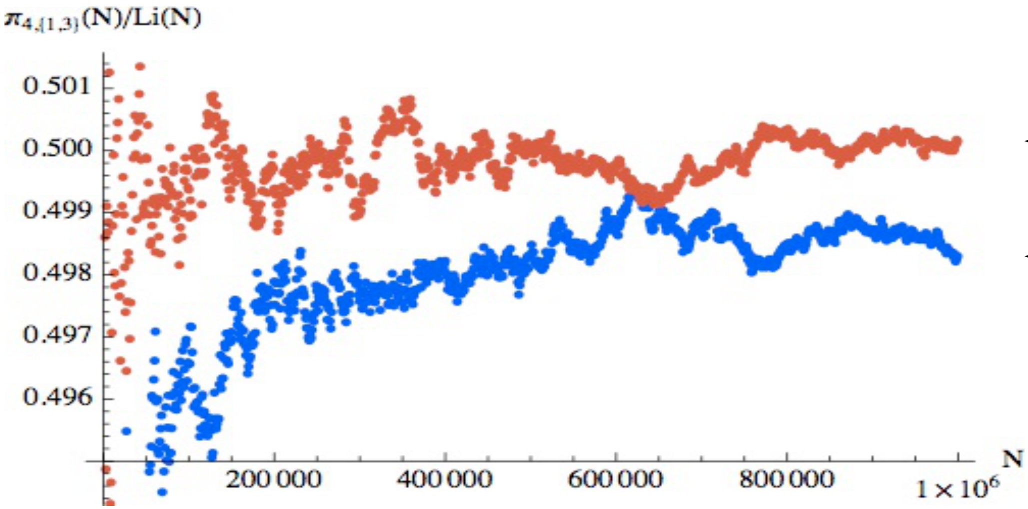
$$\pi_{4,3} : 3, 7, 11, \ldots 4n+3$$

Dirichlet theorem:

There infinite number of primes of the form 1 + 4n and 3 + 4n

PNT for arithmetic series

$$\lim_{x\to\infty}\frac{\pi_{4,1}(x)}{Li(x)} = \lim_{x\to\infty}\frac{\pi_{4,3}(x)}{Li(x)} = \frac{1}{\phi(4)} = \frac{1}{2}$$



3 mod 4

1 mod 4

Chebyshev bias:

For low values of x there are more primes 3 mod 4 than 1 mod 4

$$\Delta(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$$

Chebyshev bias gives the magnetization of qubit i=1

Twin primes (p, p+2)

Twin primes

$$\pi_2^{(1)} : (5,7),\ldots \quad (1 \bmod 4,\ 3 \bmod 4)$$

$$\pi_2^{(3)} : (11,13),\ldots \quad (3 \bmod 4,\ 1 \bmod 4)$$

Can be measured by off diagonal correlations

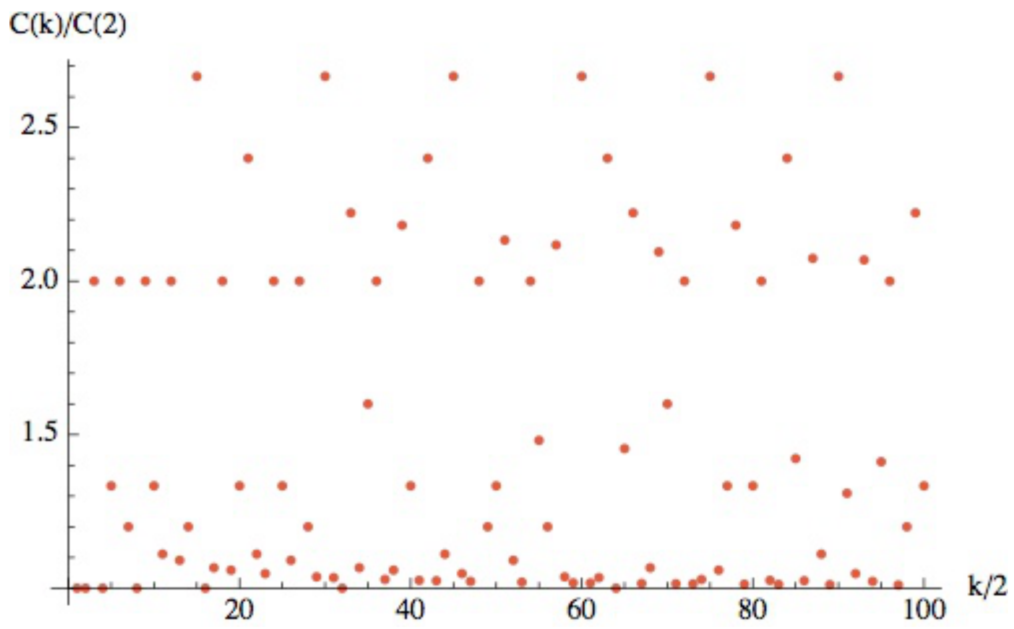
$$\langle \sigma_x^{(1)} \rangle = \frac{2\pi_2^{(1)}(2^n)}{\pi(2^n)}, \qquad \langle \sigma_x^{(1)} \sigma_x^{(2)} + \sigma_y^{(1)} \sigma_y^{(2)} \rangle = \frac{4\pi_2^{(3)}(2^n)}{\pi(2^n)}$$

Twinship → off diagonal entries of density matrix

Sub-series of primes, twin primes, etc. are amenable to measurements

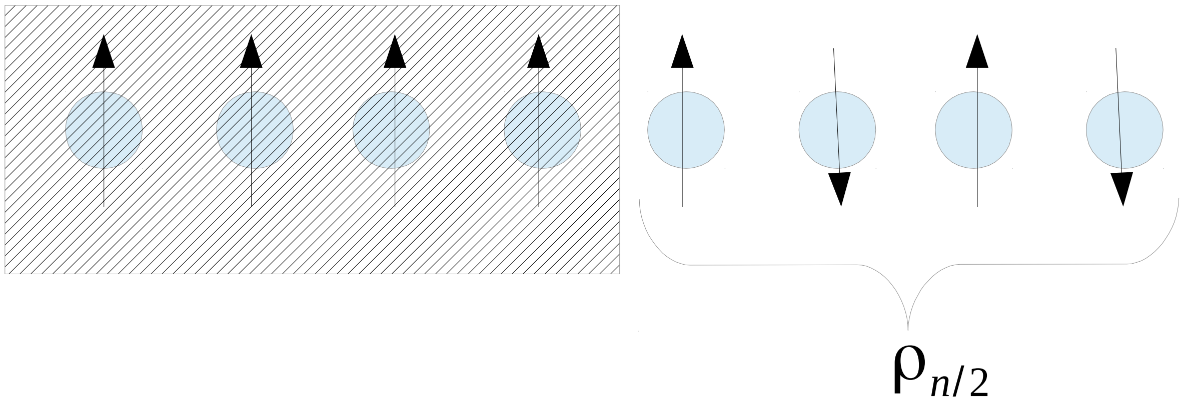
Pairwise correlations between primes (Hardy-Littlewood conjecture)

Counting p, p+k primes < x:



Twin prime constant

Reduced Density matrix



Hardy-Littlewood constant

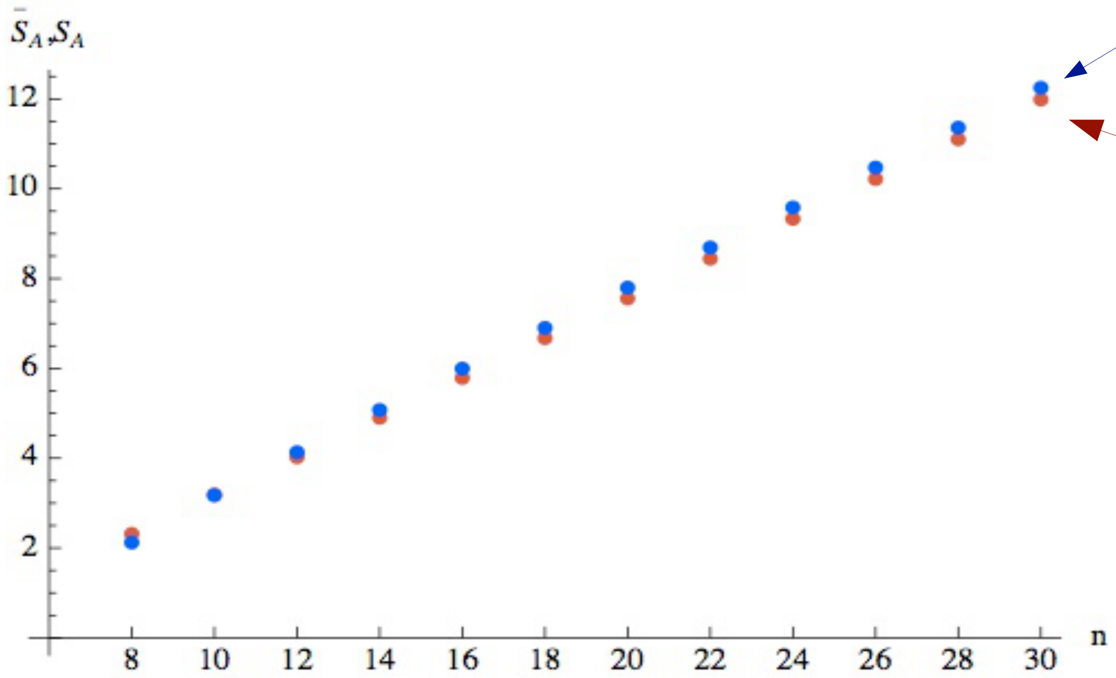
Toeplitz matrix

If primes were uncorrelated

Prime correlations :

GUE statistics of Riemann zeros (Berry, Keating, ..)

Von Neumann entropy of

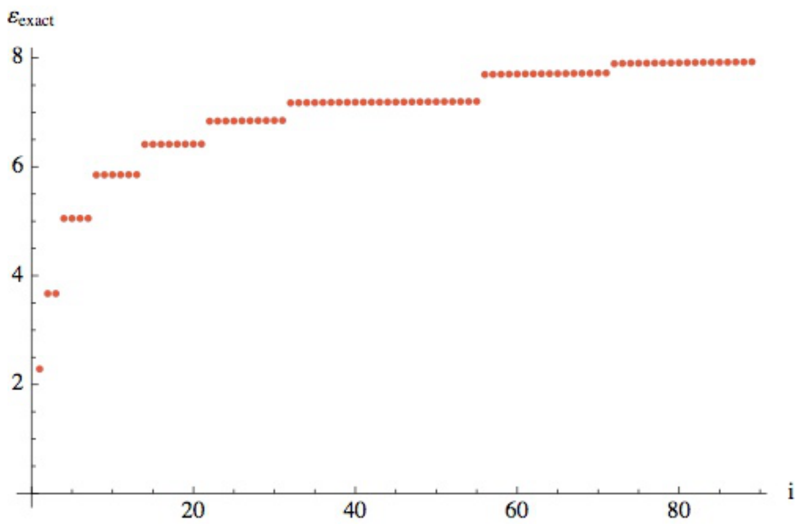


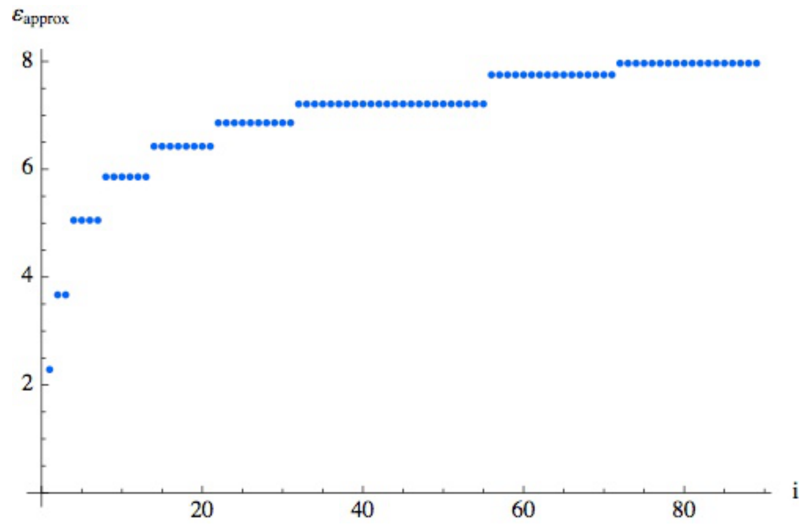
Approximation

Exact

Volume law scaling

Entanglement spectrum





$$\rho_{n/2} \sim 1 + \frac{1}{n \log 2} C_{n/2} \quad (n \to \infty)$$

Positive eigenvalues of C matrix

Analytic estimation of entanglement entropy

Renyi entropies can be related to the zeta function

Entanglement properties of the Prime state come from the pairwise
Correlations between primes (Hardy-Littlewood conjecture)

Scaling of entanglement entropy

Random states

Prime state

Area law in d-dimensions

Critical scaling in d=1
at quantum phase transitions

Finitely correlated states
away from criticality

Conclusions

- The Prime state provides a link between Number Theory and Quantum Mechanics

- Quantum Computers could be used as Quantum Simulators of Arithmetics

- Arithmetic properties could be measured more efficiently than with classical algorithms, e.g. to falsify the Riemann hypothesis

- Entanglement in the Prime state captures fundamental properties as pair correlations between the primes

- Possible connections with Random Matrix Theory, Quantum Chaos and the Riemann zeros

Thank you