Entanglement Entropy in Non Unitary CFT

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Outline

- Introduction: Von Neumann and Renyi entropies as a "measure" of Entanglement
- Entanglement entropy in 1D spin chains and in 2D CFT
- Non unitary CFT models and EE
- Corner Transfer Matrix (CTM) method
- Entanglement entropy in spin chains related to ABF and FB models
- Numerical results

Introduction

Entanglement: fundamental quantum property

Different reasons for interest:

1. Quantum Information, Quantum computers
2. Telecommunication and Teleportation
3. Black holes, Information paradox & Quantum Gravity
4. Condensed matter physics $\longrightarrow$ non-local correlations
5. Universality in Quantum Fluctuations and Phase Transitions
6. **NON LOCALITY** intrinsic in Quantum Mechanics?

- **EPR paradox** (1935): uncompleteness of QM or non-locality?
- **Bell inequalities** (1962)$\longrightarrow$ local hidden variables exist only if a certain correlation $\mathcal{P} < 2$
- Clauser Friedmann (1966) & Aspect (1980) **experiments** $\longrightarrow$ $\mathcal{P} > 2 \Longrightarrow$ possible non-locality of QM

Entanglement and density matrix

Consider a system divided in two complementary subsystems A and B

- Define **reduced density matrix** for subsystem A

$$\rho_A = \mathrm{Tr}_B |0\rangle\langle 0|$$

Quantum entropy (Von Neumann) of Entanglement (E-Entropy)

$$S_A = -\mathrm{Tr}_A(\rho_A \log \rho_A) = S_B$$

[Bennett, Bernstein, Popescu, Schumacher (1996)]

For a separable state $S_A = 0$, for a maximally entangled state it is maximal $\Longrightarrow$ S_A is a measure of Entanglement

- Area law [Srednicki (1993)]

$$S_A \propto \mathrm{Area}(\partial A)$$

Rényi Entropy

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}_A \rho_A^n \qquad \Longrightarrow \qquad S_A = S_1 = \lim_{n \to 1} S_n$$

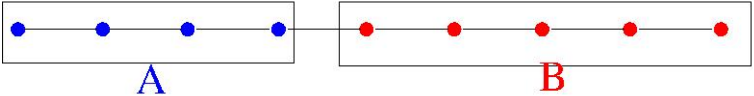
Entanglement in a Spin Chain

- Hamiltonian of a chain of length L

$$H = \sum_{k=1}^{L} H_{k,k+1}$$

- Block of spins in the space interval $[1, \ell]$ is subsystem A
- The rest is subsystem B

$\Longrightarrow$ Entanglement of a block of spins in the space interval $[1, \ell]$ with the rest of the ground state as a function of ℓ



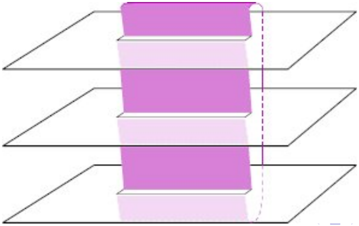
Entanglement entropy in CFT

- If the chain is **critical**, use CFT [Holzhey, Larsen, Wilczek 1994 - Calabrese, Cardy 2004]
- Partiton function of a theory with Lagrangean $\mathcal{L}$ on a Riemann surface $\mathcal{R}$= n sheets sewn by the segment $[a, b]$. It has zero curvature but for points a, b with conical singularities

$$Z[\mathcal{L}, \mathcal{R}] = \int \mathcal{D}\phi \exp \left[- \int_{\mathcal{R}} dxdy \mathcal{L}[\phi](x, y) \right]$$

- n copies of the theory

$$Z[\mathcal{L}, \mathcal{R}] = \int \mathcal{D}\phi_1 ... \mathcal{D}\phi_n e^{-\int_{\mathbb{C}} dxdy \{\mathcal{L}[\phi_1] + ... + \mathcal{L}[\phi_n]\}} = Z[\mathcal{L}^{(n)}, \mathbb{C}] \equiv Z_n$$



Twist fields (I)

- Path integral with b.c.

$$\phi_i(x, 0^+) = \phi_{i+1}(x, 0^-) \qquad , \qquad x \in [a, b]$$

- Twist fields

$$\begin{aligned} \phi_i(y)\mathcal{T}(x) &= \Theta(x_1 - y_1)\mathcal{T}(x)\phi_{i+1}(y) + \Theta(y_1 - x_1)\mathcal{T}(x)\phi_i(y) \\ \phi_i(y)\tilde{\mathcal{T}}(x) &= \Theta(x_1 - y_1)\tilde{\mathcal{T}}(x)\phi_{i-1}(y) + \Theta(y_1 - x_1)\tilde{\mathcal{T}}(x)\phi_i(y) \end{aligned}$$

- Orbifold construction [Knizhnik (1987)]

$$\begin{aligned} Z_n &\propto \langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}} \\ \langle \mathcal{O}(x)\rangle_{\mathcal{L},\mathcal{R}} &= \frac{\langle \mathcal{O}(x)\mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}} \end{aligned}$$

Twist fields (II)

- Conformal transformation

$$w \in \mathcal{R} \mapsto z \in \mathbb{C} \quad : \quad z = \left(\frac{w-a}{w-b}\right)^{\frac{1}{n}}$$

- Stress-energy tensor $T^{(n)}$ of replica theory

$$T^{(n)}(z) = \sum_{j=1}^{n} T_j(z) \quad \text{transforms as} \quad T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z, w\}$$

- 1-pt function $\langle T(w)\rangle_{\mathcal{L},\mathcal{R}}$

$$\frac{c}{24}\left(1-\frac{1}{n^2}\right)\frac{(a-b)^2}{(w-a)^2(w-b)^2} = \frac{\langle T(x)\mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}{\langle \mathcal{T}(a,0)\tilde{\mathcal{T}}(b,0)\rangle_{\mathcal{L}^{(n)},\mathbb{C}}}$$

- Comparing, get the conformal dimensions of the twist fields

$$\boxed{\Delta_n = \tilde{\Delta}_n = \frac{c}{12}\left(n - \frac{1}{n}\right)}$$

- General definition off-criticality [Cardy, Castro-Alvaredo, Doyon (2008)]

Density matrix

- Density matrix of the vacuum (ground state $|\Omega\rangle$, not to be confused with conformal vacumm $|0\rangle$)

$$\rho = |\Omega\rangle\langle\Omega|$$

- Reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho$$

- Traces

$$\mathrm{Tr}_A \rho_A^n \propto Z_n \qquad \text{normalized} \qquad \boxed{\mathrm{Tr}_A \hat{\rho}_A^n = \frac{Z_n}{Z_1^n}}$$

- Renyi (S_n) & Von Neumann (S_1) entropies

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}_A \hat{\rho}_A^n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n} \qquad , \qquad S_1 = \lim_{n \to 1} S_n$$

EE in CFT - Results

- Thermodynamic limit $L \to \infty$

$$S(\ell) \underset{\ell\to\infty}{\sim} \frac{c}{3} \log \ell + O(1)$$

c = central charge of CFT, $O(1)$ = non-universal

- Obtain results for L finite through conformal map plane $\to$ strip

$$S(\ell, L) = \frac{c}{3} \log \left(\frac{L}{\pi} \sin \frac{\ell\pi}{L} \right) + O(1)$$

$O(1)$ does not depend on ℓ/L.

- **Off-criticality** S is finite for $\ell, L \to \infty$ and computable exactly in integrable spin chains through CTM approach.

$$S(\xi) \underset{\xi\to\infty}{\sim} \frac{c}{3} \log \xi + O(1)$$

Non-unitary models

Free energy ($\beta = 1/kT$) [Affleck; Blote, Cardy, Nightingale (1986)]

$$F(\beta) = \underbrace{f\, L\beta}_{\text{bulk}} + \underbrace{\tilde{f}\, \beta}_{\text{boundary}} - \underbrace{\frac{\pi c}{6\beta}}_{\text{Casimir}} + \ldots$$

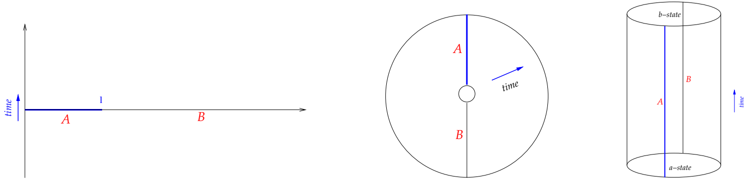
For non-unitary models [Itzykson, Saleur, Zuber (1986)]

$$c \mapsto c_{eff} = c - 24\Delta_{min}$$

Is it true also for EE?

EE in non-unitary CFT

- $\mathrm{Tr}_A(\rho_A^n)$ in the vacuum in a critical chain with boundary. Z_n =orbifold on the half-plane



- Exchange role of time and space, then transform to the cylinder

$$z \mapsto w = i \log \frac{\ell - z}{\ell + z}$$

$$Z_n = \langle a | e^{-\log \frac{\ell}{\varepsilon} \cdot H_{\mathbf{orb}}} | b \rangle \qquad , \qquad Z_1^n = \langle a | e^{-\log \frac{\ell}{\varepsilon} \cdot H_{\mathbf{rep}}} | b \rangle$$

Stress-energy tensor

$$T(z) = \sum_{k\in\mathbb{Z}} \frac{L_k}{z^{k+2}} = \sum_{j=1}^{n} T^{(j)}(z) \qquad \Longrightarrow \qquad H = L_0 + \bar{L}_0 - \frac{c}{12}$$

$$T^{(j)}(x+2\pi) = T^{(j+1)}(x) \qquad \text{orbifold (cyclic)}$$
$$T^{(j)}(x+2\pi) = T^{(j)}(x) \qquad \text{replica (periodic)}$$

- **replica**: n commuting $\mathbf{Vir}_c$: $L_k^{(j)}$, $k \in \mathbb{Z}$

$$L_k^{rep} = \sum_{j=1}^{n} L_k^{(j)} \in \mathbf{Vir}_{nc} \qquad \Longrightarrow \qquad H_{rep} = L_0^{rep} + \bar{L}_0^{rep} - \frac{nc}{12}$$

- **orbifold**: $T_{orb}(x) = T_{\lfloor \frac{x}{2\pi} \rfloor}(x \mod 2\pi) \qquad x \in [0, 2\pi n[$

$$T_{orb}(x) = \sum_{k\in\mathbb{Z}} \frac{\mathcal{L}_k}{z^{k+2}} \qquad \text{with} \qquad \mathcal{L}_k \in \mathbf{Vir}_c \quad k \in \mathbb{Z}$$

$$T(x) = \sum_{j=1}^{n} T_{orb}(x + 2\pi j) \qquad \text{has modes} \qquad \mathcal{L}_{nk}\,,\ k \in \mathbb{Z}$$

Orbifold / (Replica)^n

- Define: [Doyon, Hoogeveen, Bernard (2013)]

$$L_k^{orb} = \frac{\mathcal{L}_{nk}}{n} + \Delta_{\mathcal{T}}\delta_{0,k} \in \mathbf{Vir}_{nc} \qquad \Longrightarrow \qquad H_{orb} = L_0^{orb} + \bar{L}_0^{orb} - \frac{nc}{12}$$

- Insert a complete set of states

$$Z_n = \langle a| e^{-\log\frac{\ell}{\varepsilon}\cdot H_{orb}} \sum_s |s\rangle\langle s|b\rangle \propto e^{-2\log\frac{\ell}{\varepsilon}\left(\Delta_{:\mathcal{T}\phi:} - \frac{nc}{12}\right)}$$

$$Z_1^n = \langle a| e^{-\log\frac{\ell}{\varepsilon}\cdot H_{rep}} \sum_s |s\rangle\langle s|b\rangle \propto e^{-2\log\frac{\ell}{\varepsilon}\left(\Delta_{min} - \frac{nc}{12}\right)}$$

$$\frac{Z_n}{Z_1^n} = \mathrm{Tr}_A \rho_A^n = \left(\frac{\varepsilon}{\ell}\right)^{\frac{c_{eff}}{12}\left(n - \frac{1}{n}\right) + \dots} \qquad \Longrightarrow \qquad S_n = \frac{c_{eff}(n+1)}{12n} \log\frac{\ell}{\varepsilon} + \dots$$

$$\boxed{S = \frac{c_{eff}}{6} \log\frac{\ell}{\varepsilon} + \dots}$$

New twist field

We have introduced a new field that acts as a twist [Castro-Alvaredo, Doyon, Levi (2012)]

$$:\mathcal{T}\phi:(x) = \lim_{\varepsilon\to 0} \varepsilon^{2\left(1-\frac{1}{n}\right)\Delta_{min}} \mathcal{T}(x+\varepsilon)\phi(x)$$

allowing to express the trace of powers of ρ in a natural way in non-unitary models where the vacuum is not the conformally invariant state $|0\rangle$, but

$$|\phi\rangle = \phi(0)|0\rangle$$

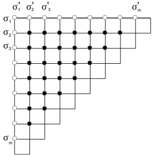
where $\phi(z)$ is the field with lower (negative) conformal dimesion Δ_{min}

$$\mathrm{Tr}_A\rho_A^n \propto \begin{cases} \frac{\langle:\mathcal{T}\phi:(\ell)\rangle}{\langle\phi(\ell)\rangle^n} & \text{on the half-plane} \\ \\ \frac{\langle:\mathcal{T}\phi:(\ell):\mathcal{T}\phi:(0)\rangle}{\langle\phi(\ell)\phi(0)\rangle^n} & \text{on the plane} \end{cases}$$

The same approach could be used also for negativity

Corner Transfer Matrix

- CTM is a very useful tool [Baxter (1972)]

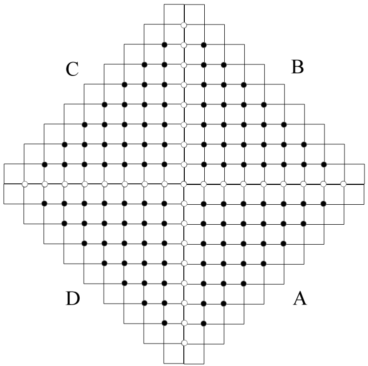


$$A_{\bar{s},\bar{s}'} = \sum_{\bullet} \prod w_i$$

- and analogously B, C, D with 90° rotations.

Partition function and CTM

- Now we can build up the whole lattice by using the 4 CTM's



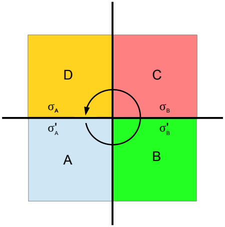
- Partition function

$$\mathcal{Z} = \sum_{\bar{\sigma},\bar{\sigma}',\bar{\sigma}'',\bar{\sigma}'''} A_{\bar{\sigma}\bar{\sigma}'} B_{\bar{\sigma}'\bar{\sigma}''} C_{\bar{\sigma}''\bar{\sigma}'''} D_{\bar{\sigma}'''\bar{\sigma}} = \mathrm{Tr}(ABCD)$$

Reduced density matrix and CTM

- Now suppose to divide the spins in two subsystems A: $\bar{\sigma}_A = (\sigma_1, ..., \sigma_p)$ and B: $\bar{\sigma}_B = (\sigma_{p+1}, ..., \sigma_L)$, i.e. $\bar{\sigma} = (\bar{\sigma}_A, \bar{\sigma}_B)$
- Reduced density matrix of subsystem A

$$\rho_A(\bar{\sigma}_A, \bar{\sigma}'_A) = \sum_{\bar{\sigma}_B} \langle \bar{\sigma}_A, \bar{\sigma}_B | 0 \rangle \langle 0 | \bar{\sigma}'_A, \bar{\sigma}_B \rangle = \mathrm{Tr}_B \langle \bar{\sigma}_A | 0 \rangle \langle 0 | \bar{\sigma}'_A \rangle$$



$$\rho_A = (ABCD)_{\bar{\sigma}, \bar{\sigma}'} \quad \Longrightarrow \quad S_n = \frac{1}{1-n} \log \mathrm{Tr}_A \rho_A^n$$

EE in FB models

- Continuum limit of ABF models on square lattice ($RSOS_m$). CTM diagonalization is given and the calculation of ρ_A has been done [Franchini, De Luca (2012)]
- Can be generalized to FB non-unitary $RSOS_{m,m'}$ models $a = 1, ..., m' - 1\,,\ d = 1, ..., m - 1$ and $t = \frac{T - T_c}{T_c}$

$$Z_n = \sum_{\substack{a=1 \\ a = \lfloor \frac{dm'}{m} \rfloor \mod 2}}^{m'-1} E(x^a, y)^n F(a, d; x^{2n}) \quad , \qquad y = e^{\frac{4\pi^2}{\log t}}, \quad x = y^{\frac{m'-m}{m'}}$$

$$E(x, y) = \sum_{n \in \mathbb{Z}} (-1)^k y^{\frac{k(k-1)}{2}} x^k \qquad F(a, d; q) = q^{\frac{(a-d)(a-d-1)}{4}} q^{\frac{c}{24} - \Delta_{da}} \chi_{da}(q)$$

- Renyi entropy

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}_A \rho_A^n = \frac{1}{1-n} \log Z_n - \frac{n}{1-n} \log Z_1$$

expanding for $t \to 0$ with $\xi \sim t^{-\nu}$, with $\nu = \frac{m'}{4(m'-m)}$

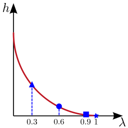
$$\boxed{S_n = \frac{(n+1)c_{eff}}{12n} \log \xi + \dots}$$

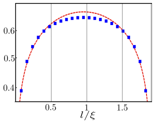
Numerical results

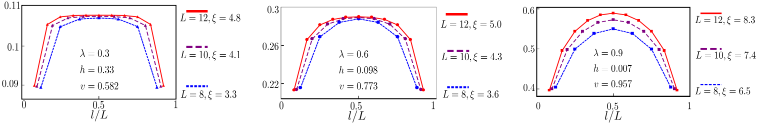
Spin chain [von Gehlen (1994)]

$$H(\lambda, h) = \frac{1}{2}\sum_{i=1}^{L}(\sigma_i^z + \lambda\sigma_i^x\sigma_{i+1}^x + ih\sigma_i^x)$$

has a critical line in the (λ, h)-plane with $c = -\frac{22}{5}$ $(c_{eff} = \frac{2}{5})$: Lee-Yang universality class.







$S = \frac{c_{eff}}{3} \log\left(\frac{L}{\pi}\sin\frac{\ell\pi}{L}\right) + \alpha$ Numerically $c_{eff} = 0.4056$ and $\alpha = 0.3952$

Quantum critical hamiltonian

- *"The answer is yes, but... what was the question?"* [W. Allen]: We know the 2D classical lattice model, we can compute formally S_n, but what is the quantum Hamiltonian we are dealing with?

- At criticality $\mathcal{U}_q(sl(2))$ invariant XXZ model [Alcaraz, Barber, Batchelor (1988) - Pasquier, Saleur (1990)]

$$H = -J\left[\sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \frac{q+q^{-1}}{2}\sigma_n^z\sigma_{n+1}^z) + \frac{q-q^{-1}}{2}(\sigma_1^z - \sigma_N^z)\right]$$

Can be rewritten in terms of Temperley-Lieb operators

$$H = -J\sum_{n=1}^{N-1} e_n$$

$$e_n^2 = -(q+q^{-1})e_n \quad , \quad e_n e_{n\pm1} e_n = e_n \quad , \quad e_n e_m = e_m e_n \text{ if } |n-m| > 1$$

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Quantum off-critical hamiltonian

What happens off-criticality? Introduce the tile operators
$(\mathbf{a} = (a_1, a_2, ..., a_N))$

$$\mathbf{1}\Big|_{\mathbf{a}}^{\mathbf{a}'} = \prod_i \delta_{a_i,a_i'}$$

$$\mathbf{e}_j\Big|_{\mathbf{a}}^{\mathbf{a}'} = \left[\left(\prod_{i\neq j} \delta_{a_i,a_i'}\right)\delta_{a_{j-1},a_{j+1}}\right]\frac{s(a_j'\lambda)}{s(a_{j+1}\lambda)} \quad , \qquad s(u) = \vartheta_1(u,t)$$

$$\mathbf{g}_j\Big|_{\mathbf{a}}^{\mathbf{a}'} = \left[\left(\prod_{i} \delta_{a_i,a_i'}\right)\delta_{a_{j-1},a_{j+1}}\right]\left[(a_j'-a_{j+1})\frac{s'(a_{j+1}\lambda)}{s(a_{j+1}\lambda)} + \frac{s'(\lambda)}{s(\lambda)} - \frac{s'(0)}{s(\lambda)}\frac{s(a_j'\lambda)}{s(a_{j+1}\lambda)}\right]$$

Hamiltonian

$$H = -\frac{d}{du}\log \mathbf{T}(u)\big|_{u=0} = -J\sum_{j=1}^{N-1}\left[\frac{s'(0)}{s(\lambda)}\mathbf{e}_j - \frac{s'(\lambda)}{s(\lambda)}\mathbf{1} + \mathbf{g}_j\right]$$

Limit $t \to 0$: $\mathbf{g}_j \to 0$ while $\mathbf{e}_j \to$TL-algebra
In general, algbera with **two** parameters ($\Longrightarrow$ elliptic algebras?)

Summary

- Von Neumann and Rényi E-Entropies are crucial tools to study entanglement in quantum systems. In integrable models, they can be calculated using integrable techniques.
- Corner Transfer Matrix technique allows the exact calculation of bipartite E-Entropy in spin chains. Having the exact formula at hand, one can test some of the open issues about entanglement in these models.
- In the case of non-unitary theories, the coefficient of the logarithmic divergence near criticality gives c_{eff} instead of c. Although this result is not surprising, it sheds more light on the general way to compute finite interval density matrices in generic CFT's.
- An integrable way to compute finite size E-Entropy is to be developed. It would complement the present knowledge by new precious information.

Conclusions

- Entanglement entropy is a new way to approach interesting problems in theoretical physics and it should be better understood in (integrable) QFT, as it seems crucial in the solution of challenging paradoxes, like the information loss in black holes.
- It also stimulates progresses in mathematics, in the best tradition of the integrability approach.

Thank you!!!

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