Workshop on

Entanglement Entropy in Many Body Quantum Systems

Universal Quantum Simulator, Local Convertibility and Edge States in Many-Body Systems

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(submitted to PRX, almost accepted)

Entanglement

- Entanglement: fundamental quantum property
- Different reasons for interest:

1. Quantum information → quantum computers
2. Quantum Phase Transitions → universality
3. Condensed matter → non-local correlator
4. Integrable Models → new playground
5. Cosmology → Black Holes
6. …

Entanglement: what is it good for?

- Characterization of quantum states and how to simulate them (DMRG, MPS…..)
- Detection of novel quantum phases (topological phases)
- Can determine computational power of a quantum phase?
- Does a quantum phase transition change such comp. power?

→ Our answer: if QPT yields degeneracy from edge states ⇒ the long-range order of these boundary states gives phase a greater quantum computational power

Understanding Entanglement

- Consider a unique (pure) ground state
- Divide system into two Subsystems: A & B
- If system wave-function:

$$|\Psi^{A,B}\rangle = |\Psi^{A}\rangle \otimes |\Psi^{B}\rangle$$ → No Entanglement

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{\mathcal{D}} \sqrt{\lambda_j} |\Psi_j^{A}\rangle \otimes |\Psi_j^{B}\rangle$$ → Entangled

(with $\mathcal{D} > 1$, $|\Psi_j^{A}\rangle$ & $|\Psi_j^{B}\rangle$ linearly independent):

- Entangled: Measurements on B affect A

Von Neumann & Renyi Entropies

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j} |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

$$\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle\langle\Psi^{A,B}| = \sum \lambda_j |\Psi_j^A\rangle \, \langle\Psi_j^A|$$

- Von Neumann (Quantum analog of Shannon Entropy):

$$S_A = -\mathrm{tr}_A \left(\rho_A \log \rho_A\right) = -\sum \lambda_j \log \lambda_j$$

- Renyi Entropy $\rightarrow$ Entanglement spectrum

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{tr} \left(\rho_A^\alpha\right) = \frac{1}{1-\alpha} \log \sum_j \lambda_j^\alpha$$

(equal to Von Neumann for $\alpha \rightarrow 1$)

- Remark: $S_B = -\mathrm{tr}_B \left(\rho_B \log \rho_B\right) = S_A$

LOCC & Entanglement

- Consider bi-partite states (A | B): $|\Psi_{A,B}\rangle$ & $|\Phi_{A,B}\rangle$

- Entanglement cannot increase under Local Operations & Classical Communications (LOCC)

$\Rightarrow$ if $S_\alpha([\Phi]) < S_\alpha([\Psi]) \quad \forall\alpha$

$|\Psi_{A,B}\rangle$ can be converted to $|\Phi_{A,B}\rangle$ but not vice-versa!

S. Turgut JPA (2007)

(Depends upon partition choice!)

- A state can only be converted to one of lower entanglement

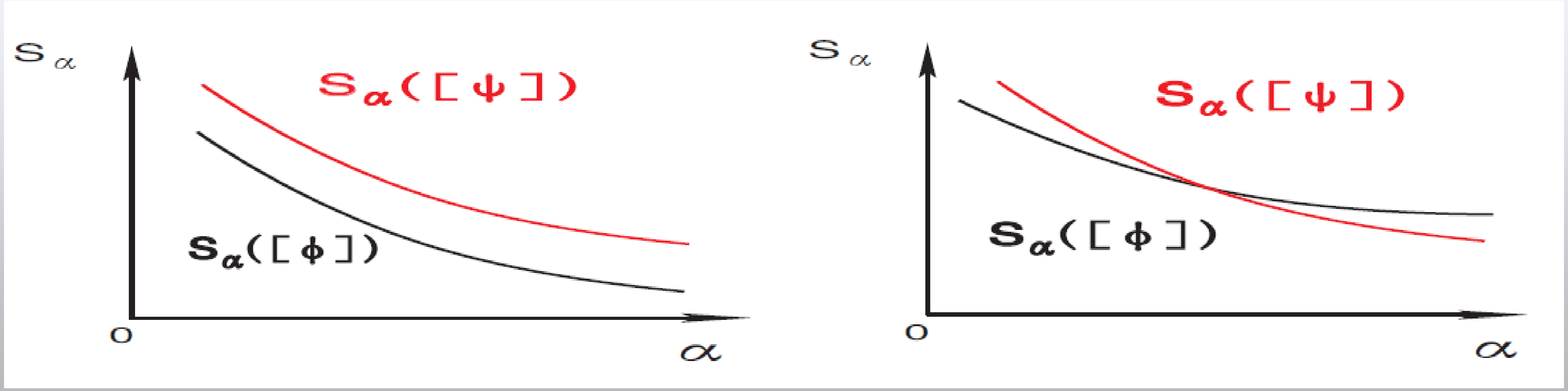
Local Convertibility

- Take two bipartite states: $|\Psi_{A,B}\rangle$ & $|\Phi_{A,B}\rangle$

- If $\exists \alpha_1$ such that $S_{\alpha_1}([\Phi]) < S_{\alpha_1}([\Psi])$ &

  $\exists \alpha_2$ such that $S_{\alpha_2}([\Phi]) > S_{\alpha_2}([\Psi])$

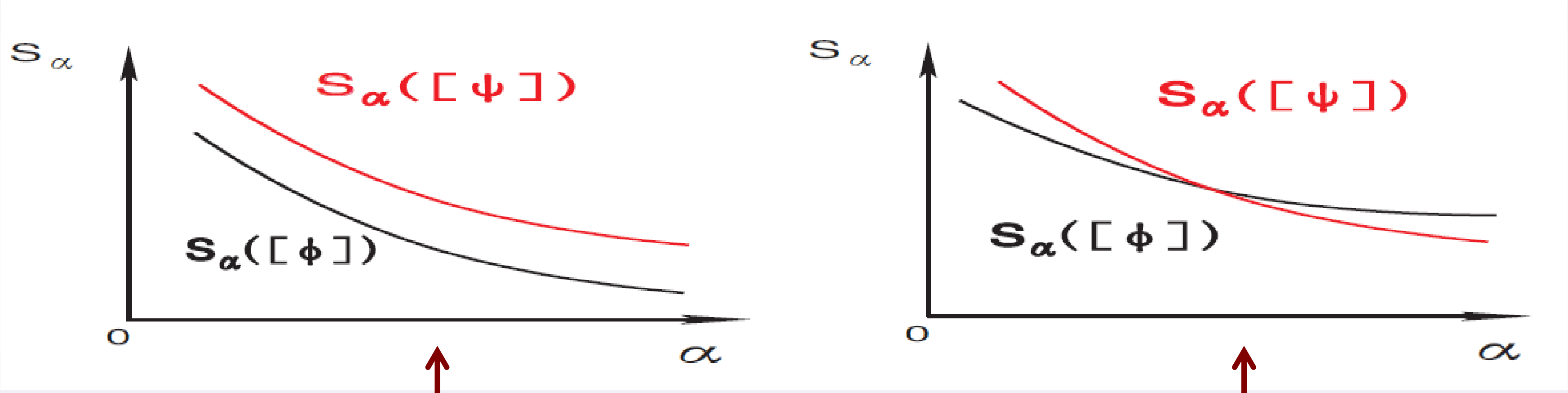
$\Rightarrow$ the two states cannot be transferred locally (by LOCC) one into the other



Local Convertibility & Adiabatic Evolution

- Adiabatic evolution: $|\Psi_{A,B}\rangle$ ground state of H(g)

  and $|\Phi_{A,B}\rangle$ ground state of H(g + Δ g)



S_α(g) monotonous

S_α(g) non-monotonous

- **Study Renyi entropy derivative w.r.t g as function of α**

  $\rightarrow$ Differential Local Convertibility

Local Convertibility & Entropy derivative

- Adiabatic evolution: Renyi entropy of instantaneous ground state of Hamiltonian H(g) as function of g and α

- If $\frac{\mathrm{d}S_\alpha}{\mathrm{d}g}$ changes sign as α varies

  $\Rightarrow$ LOCC cannot simulate evolution

Sign of entropy derivative distinguishes computational power of different phases

Field Theory / Universality

- Naively, we expect all entanglement entropies to increase with the correlation length

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \sqrt{\lambda_j} |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle \longrightarrow \rho_A = \sum \lambda_j |\Psi_j^A\rangle \langle \Psi_j^A|$$

$$S_\alpha = \frac{1}{1-\alpha} \log \mathrm{tr}\left(\rho_A^\alpha\right) = \frac{1}{1-\alpha} \log \sum_j \lambda_j^\alpha$$

- Approaching a QPT, scale invariance require more eigenvalues to contribute equaly:

$$\mathrm{Tr}\rho_A = \sum_{j=1}^{\mathcal{D}} \lambda_j = 1 \,, \quad \rightarrow \quad \lambda_j \simeq \frac{1}{\mathcal{D}}$$

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ARTICLE

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Quantum phases with differing computational power

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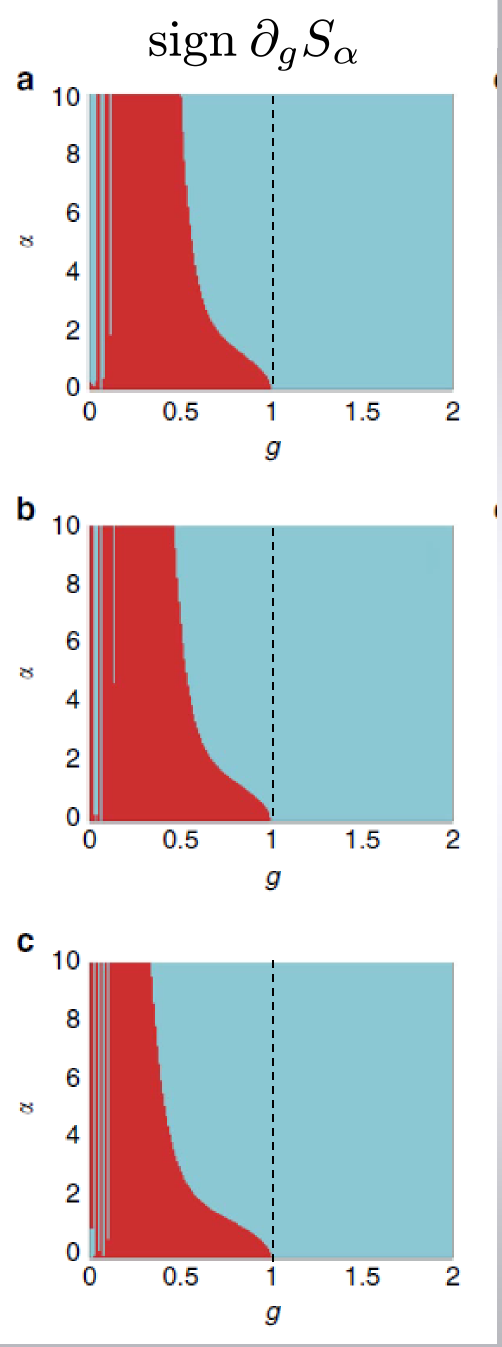
Cui et al. – Nature Comm. (2012)
Numerical Results

$$H_I = -\sum_{j=1}^{N} \left(\sigma_j^x \sigma_{j+1}^x + g\, \sigma_j^z \right)$$

- Ising model for N=12 and bipartitions (6|6), (7|5), (8|4)
- Sign of entropy derivative:
  Blue = Negative; Red = Positive
- Ferromagnetic phase more powerful for adiabatic quantum computation!
- Not true for large subsystems!



Local Convertibility & Topological Order

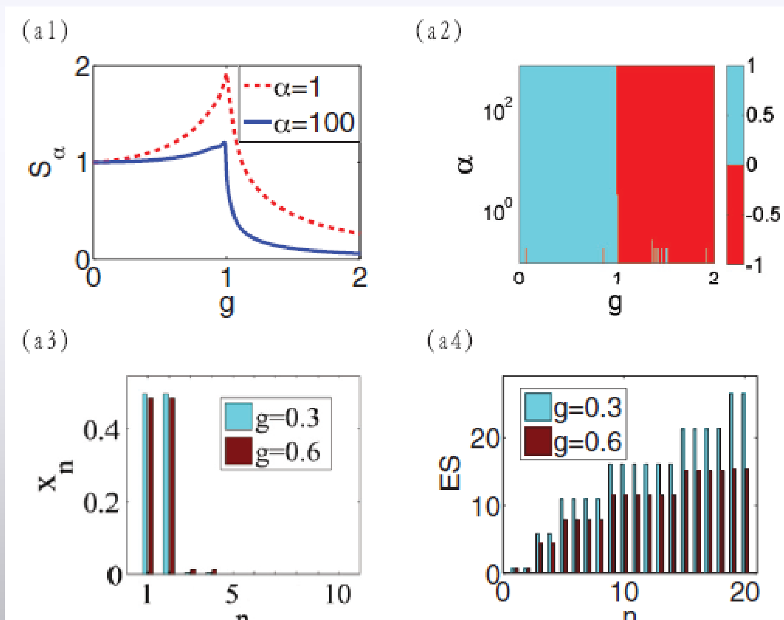
PHYSICAL REVIEW B **88**, 125117 (2013)

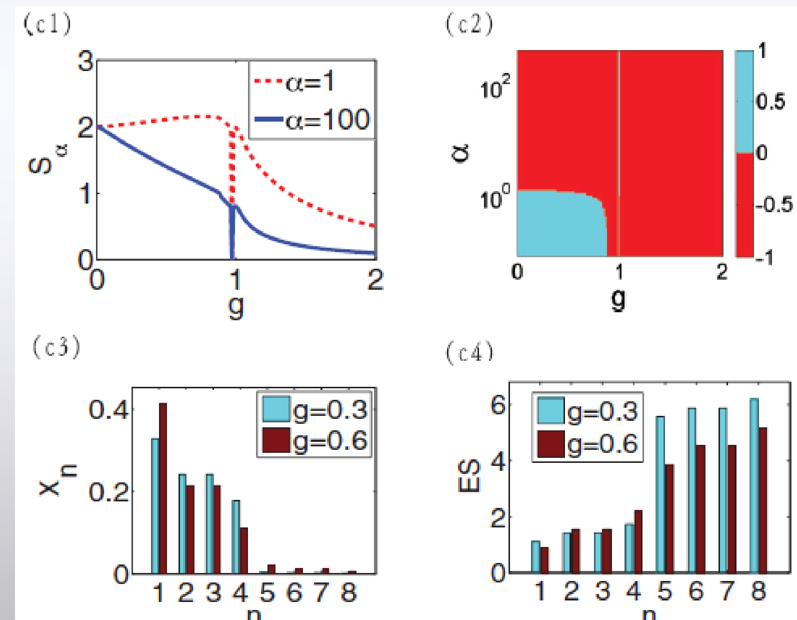
Local characterization of one-dimensional topologically ordered states

Jian Cui,[1,2] Luigi Amico,[3,4] Heng Fan,[1] Mile Gu,[4,5] Alioscia Hamma,[5,6] and Vlatko Vedral[4,7,8]

- *Cluster Ising Model:*

$$H(g) = -\sum_{j=1}^{N} \sigma_{j-1}^{x}\sigma_{j}^{z}\sigma_{j+1}^{x} + g\sum_{j=1}^{N} \sigma_{j}^{y}\sigma_{j+1}^{y}$$





(50|50) **(48|3|49)**

Local Convertibility & Topological Order

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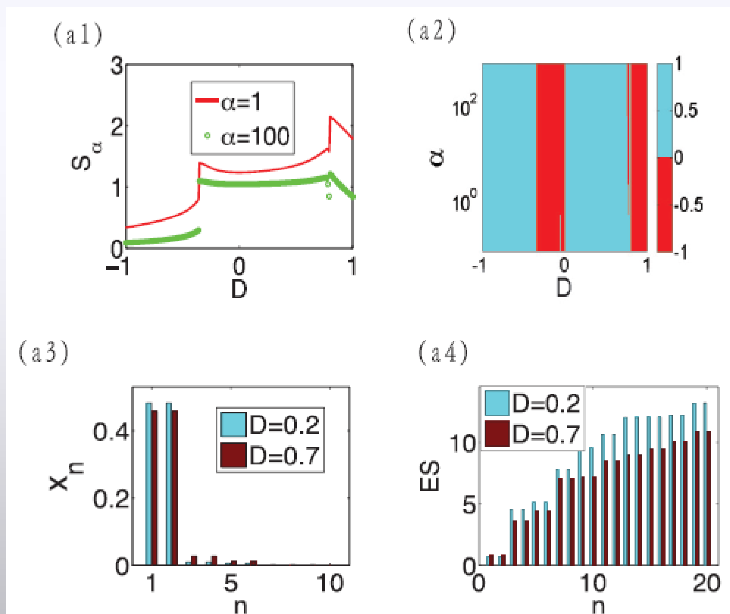
- **The λ-D Model:**

$$H = \sum_i \left[\left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + \lambda S_i^z S_{i+1}^z + D \left(S_i^z \right)^2 \right]$$



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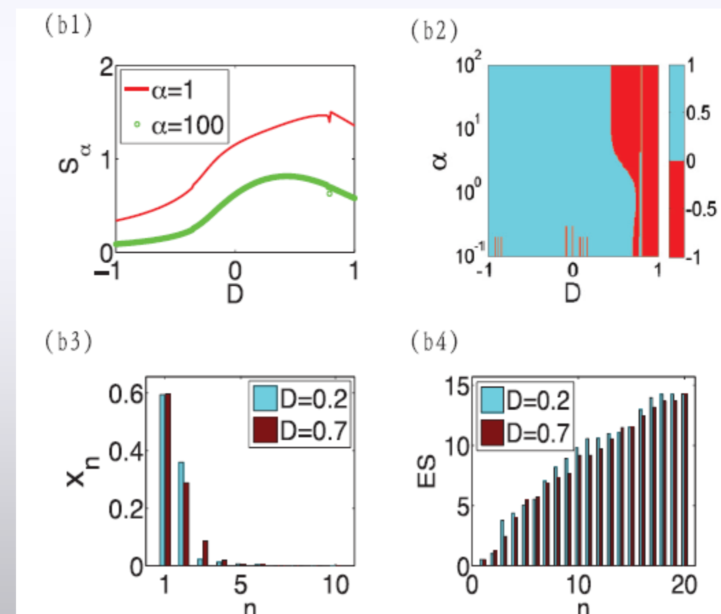

(96|4)

Local Convertibility & Topological Order

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Local Response of Topological Order to an External Perturbation

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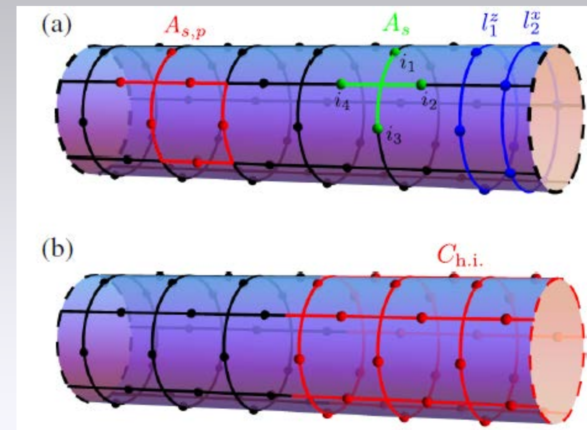
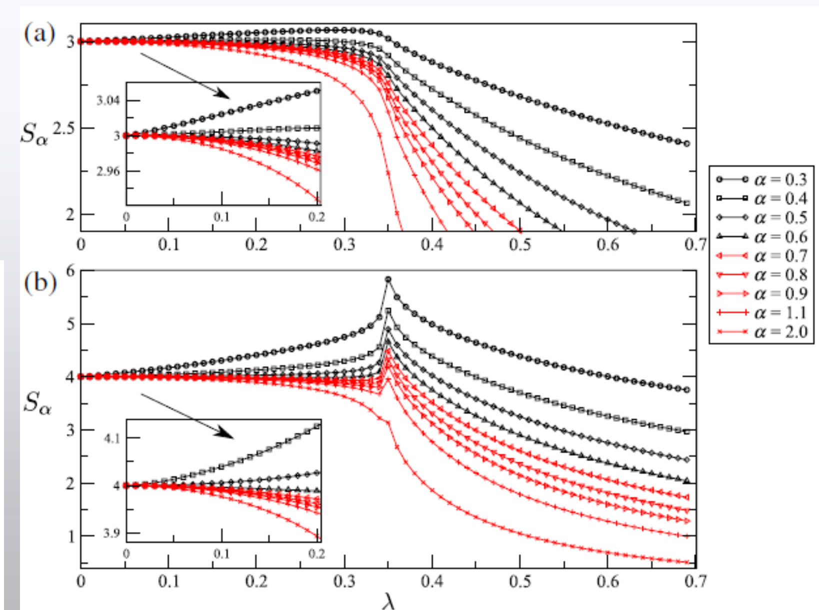


- Perturbed 2-D Toric Code:

$$\mathcal{H} = -\sum_s \prod_{i\in s} \sigma_i^x - \sum_p \prod_{i\in p} \sigma_i^z + V(\lambda)$$

Perturbation $V(\lambda)$	G.I.	DLC	Exact	ξ
$\sum_s e^{-\lambda_s \sum_{i\in s}\sigma_i^z}$	✓	✓	✓	0
$\lambda_h \sum_{i\in H}\sigma_i^z$	✓	×	✓	$\neq 0$
$\lambda_z \sum_i \sigma_i^z$	✓	×	×	$\neq 0$
$\lambda_z \sum_i \sigma_i^z + \lambda_x \sum_j \sigma_j^x$	×	×	×	$\neq 0$



The Quantum Ising Chain

$$H_I = -\sum_{j=1}^{N} \left(t\, \sigma_j^x \sigma_{j+1}^x + h\, \sigma_j^z \right)$$

$\downarrow$

Exact (non-local) mapping into free fermions

(Jordan-Wigner + Bogolioubov rotation)

$\downarrow$

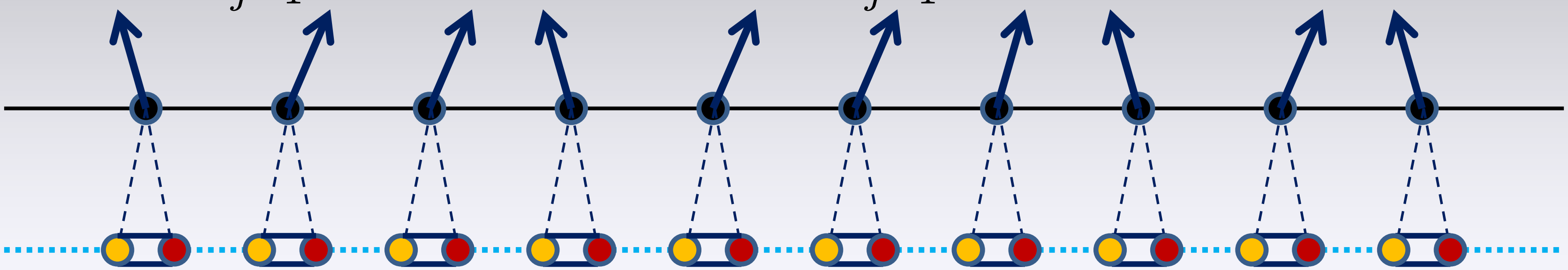
$$H_I = \sum_q \varepsilon_q \left(\chi_q^\dagger \chi_q - \frac{1}{2} \right) , \qquad \varepsilon_q = \sqrt{t^2 + h^2 - 2ht \cos q}$$

- $h/t > 1 \rightarrow \langle \sigma^x \rangle = 0$ — Paramagnetic phase
- $h/t < 1 \rightarrow \langle \sigma^x \rangle \neq 0$ — Ferromagnetic phase
- $h/t = 1$ — Ising QPT: c=1/2

Kitaev Chain

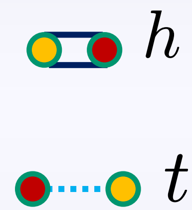
Kitaev (2001)

$$H_I = -\sum_{j=1}^{N} \left(t\, \sigma_j^x \sigma_{j+1}^x + h\, \sigma_j^z \right) = -\sum_{j=1}^{N} \left(t\, f_j^{(2)} f_{j+1}^{(1)} + h\, f_j^{(1)} f_j^{(2)} \right)$$



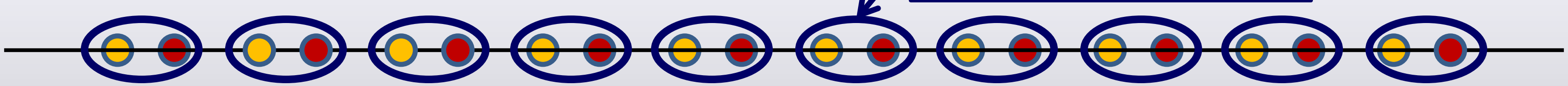
Majorana Fermion $f_j^{(1)} \equiv \sigma_j^x \prod_{l<j} \sigma_l^z$

Majorana Fermion $f_j^{(2)} \equiv \sigma_j^y \prod_{l<j} \sigma_l^z$

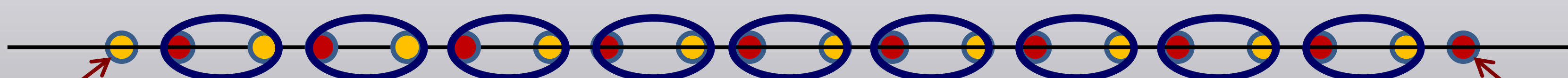


$h/t \to \infty$

$c_j = f_j^{(1)} + i f_j^{(2)}$



$h/t \to 0$



Edge State

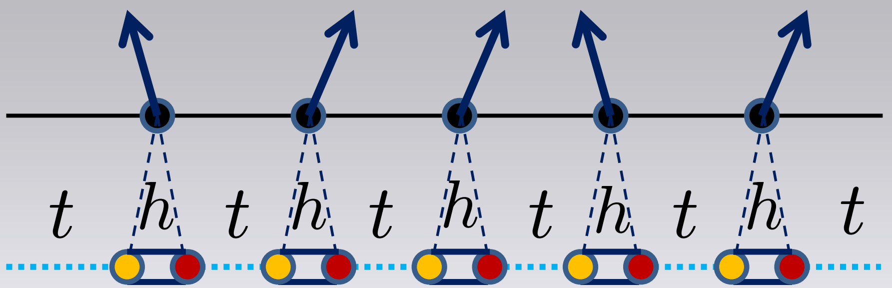
$c_j = f_{j+1}^{(1)} + i f_j^{(2)}$

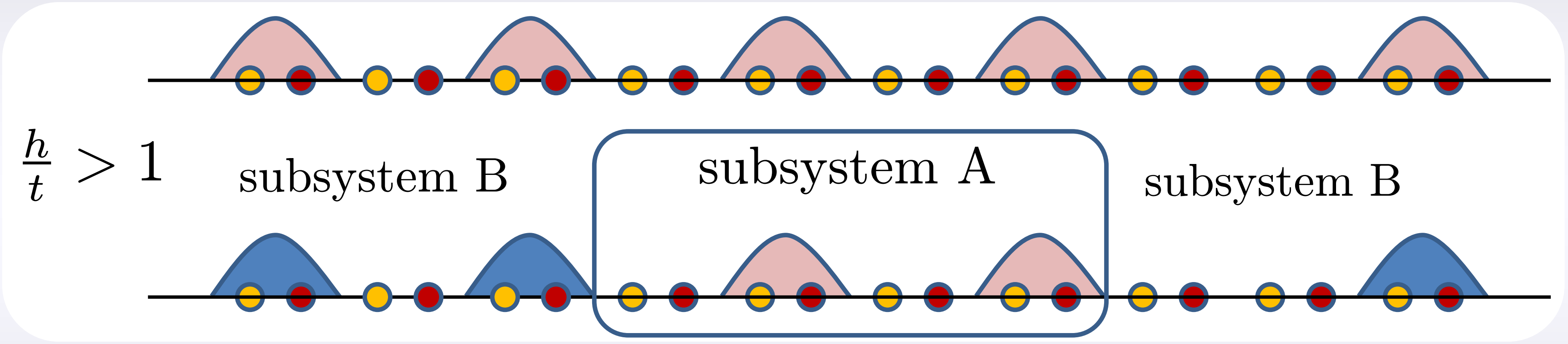
Edge State

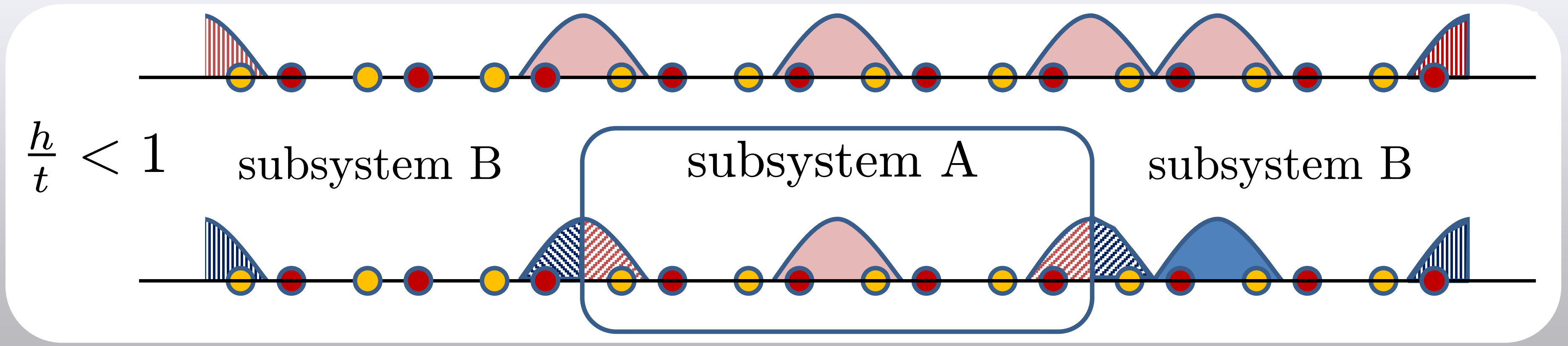
Edge States

$$H_I = -\sum_{j=1}^{N}\left(t\,\sigma_j^x\sigma_{j+1}^x + h\,\sigma_j^z\right)$$

$$= -\sum_{j=1}^{N}\left(t\,f_j^{(2)}f_{j+1}^{(1)} + h\,f_j^{(1)}f_j^{(2)}\right)$$







Edge States Entanglement

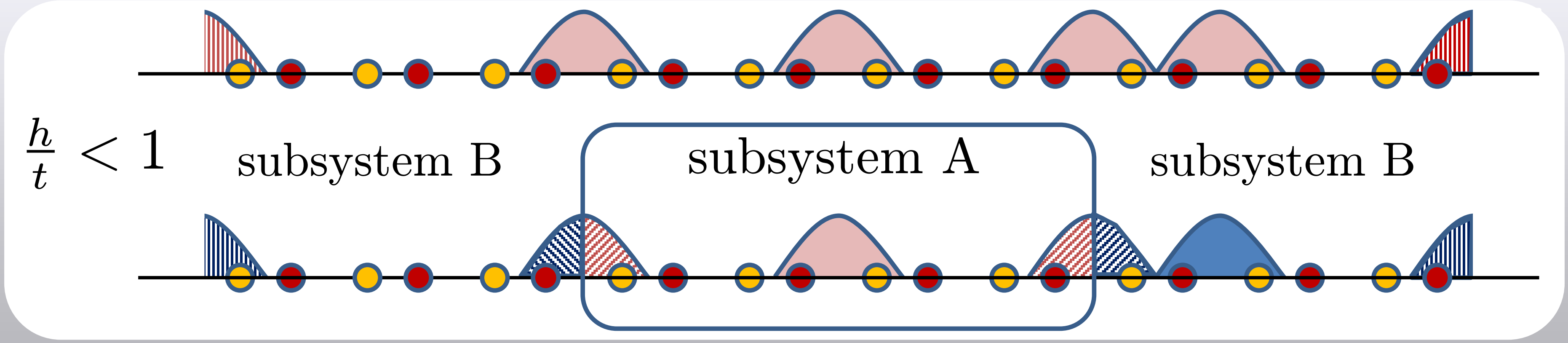
- Edge states combined into a complex fermion:

  occupied/empty $\Rightarrow$ two-fold degeneracy

  $\rightarrow$ Long-range entanglement among edge states

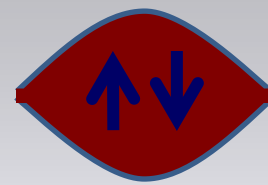
- Edge states also generated by partitioning

- Grow closer as correlation length increases



EPR Analogy

$S = 0$

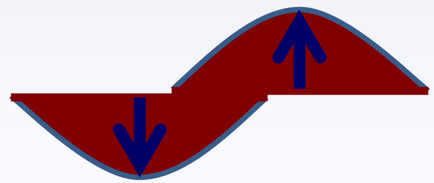


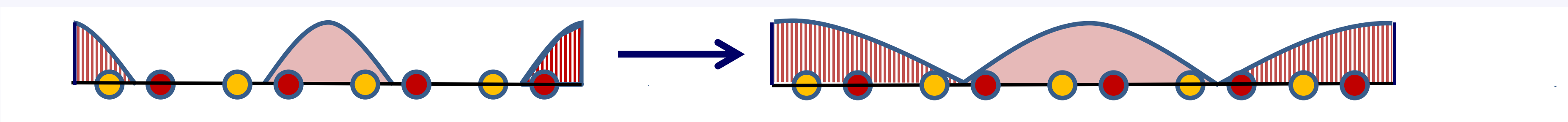
$|0\rangle$

$S = \ln 2$

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

$0 < S < \ln 2$





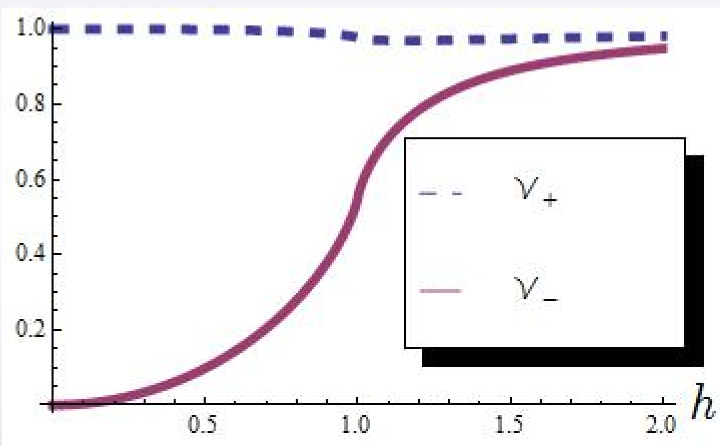
- Approaching the QPT, edge states effectively grow closer

$\Rightarrow$ their entanglement can decrease
(while bulk states entanglement increases)

Occupation Numbers

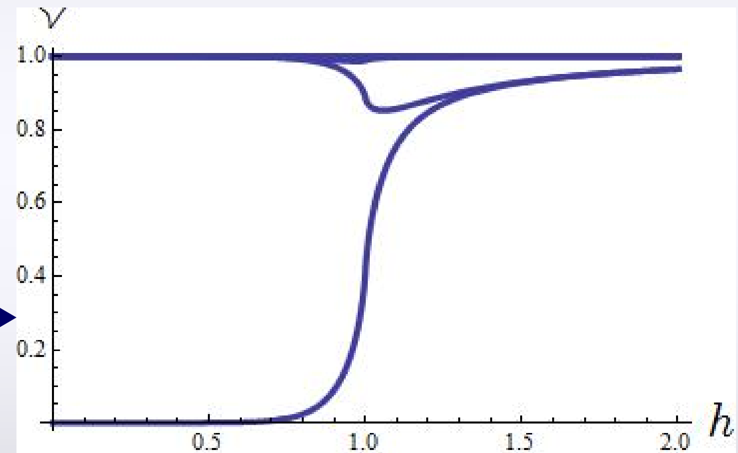
- Hilbert space of the A block from individual excitations
- Calculate their occupation number on ground state:

$$\langle 0|n_j|0\rangle = \frac{1+\nu_j}{2}$$



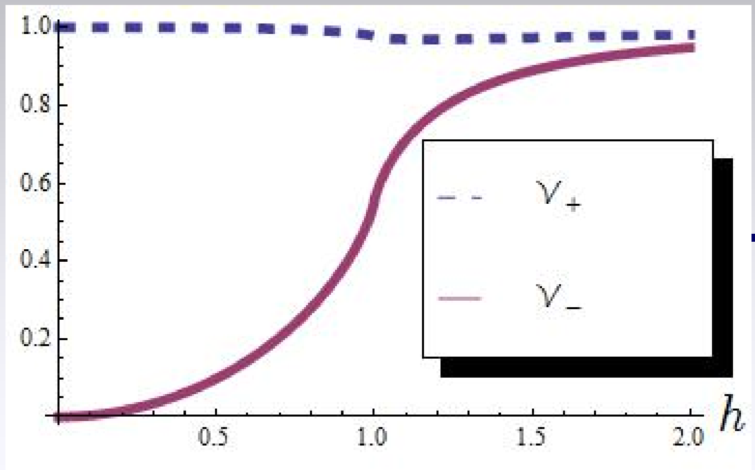
L=2

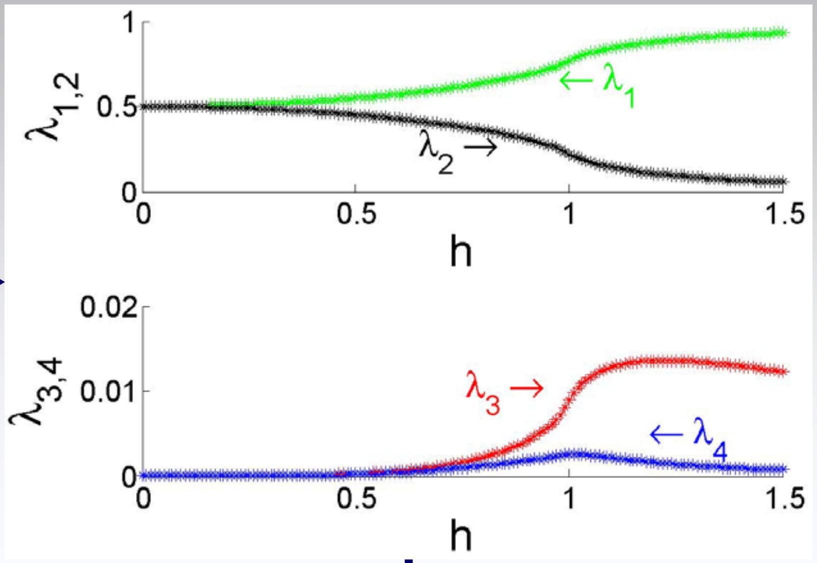
L=10

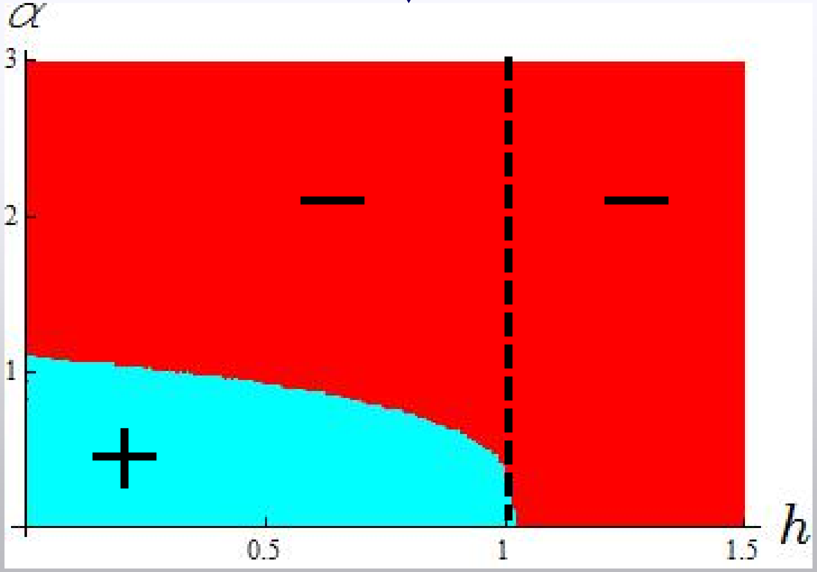


- One edge state for h<1: partial overlap
- Approaching QPT: bulk states overlap decreases,
  edge states overlap increases

2-Sites Block Entanglement



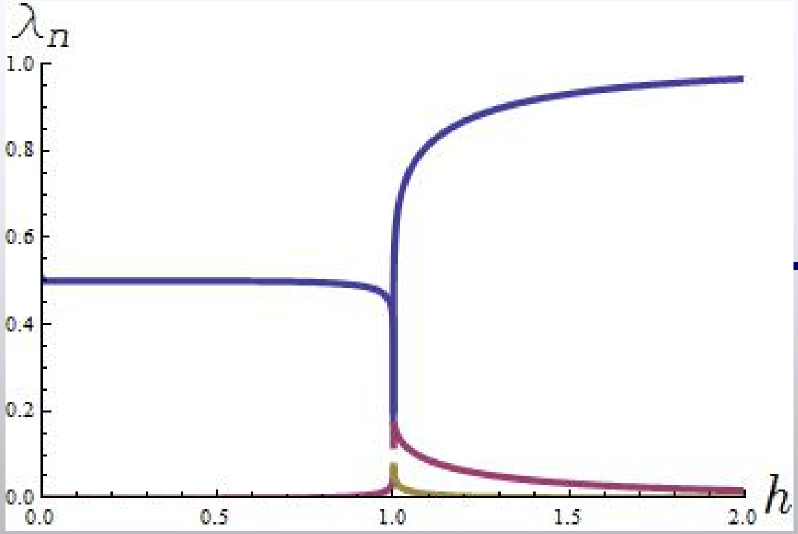


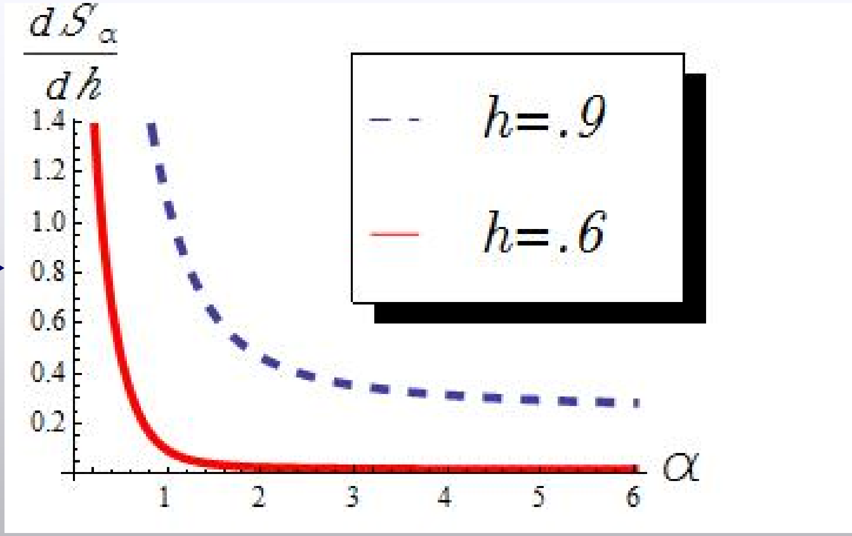


- Lack of local convertibility due to edge state recombination
- 2-sites classical gates destroy long-range correlations!

Large Block Entanglement

- For $L \to \infty$ we have full analytical knowledge of entanglement (spectrum) Its & al. (2005); F.F. & al. (2008); F.F & al. (2011)
- For $h/t < 1$ edge states give double degeneracy
- Local convertibility restored!
- Numerics confirm





Z_2 Symmetry

$$H_I = -\sum_{j=1}^{N} \left(t\,\sigma_j^x \sigma_{j+1}^x + h\,\sigma_j^z \right)$$

- Ising model: prototype of $\mathbb{Z}_2$ symmetry
- Realized non-locally: string order parameter: $\mu_N^x = \prod_{j=1}^{N} \sigma_j^z$
- Eigenstates with $\mathbb{Z}_2$ symmetry: $\langle \sigma^x \rangle = 0$

  $\rightarrow$ thermal ground state (no edge states)

- Symmetry broken states: $\langle \sigma^x \rangle \neq 0$

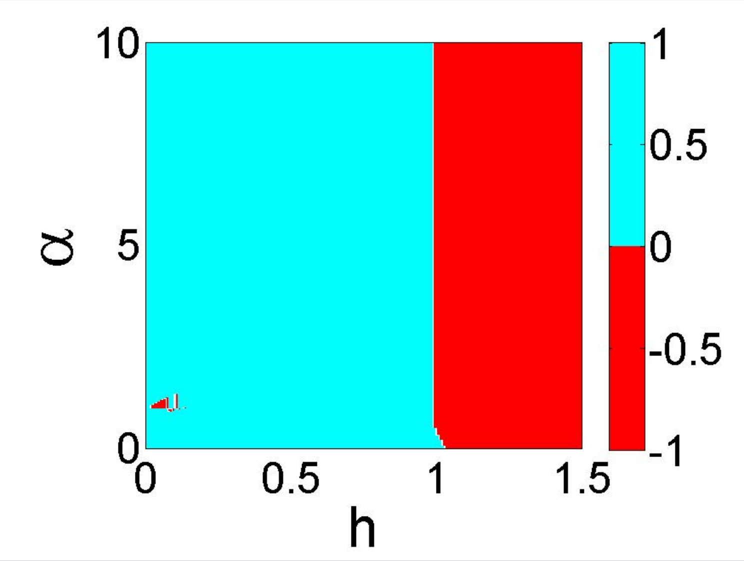
  (edge states!)

Symmetry broken Ground State

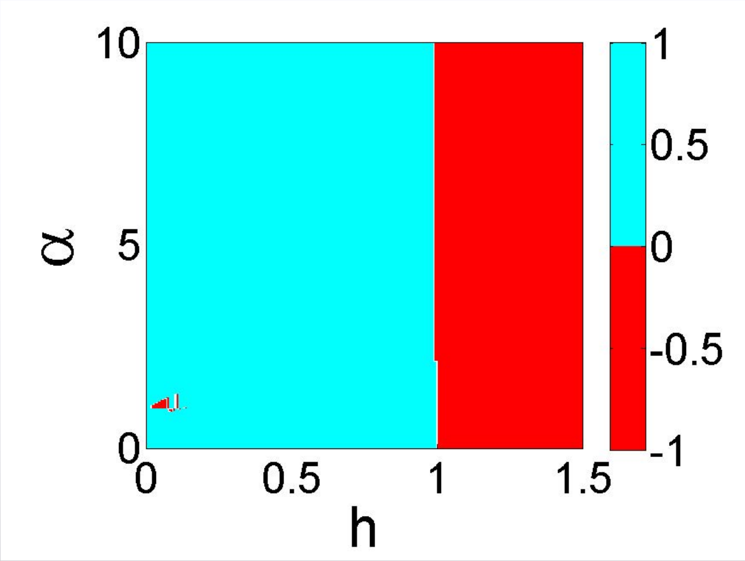
- For $h < 1$, $\langle \sigma^x \rangle \neq 0$: symmetry broken state

  $\rightarrow$ no edge states $\rightarrow$ locally convertible!

- No analytical approach, just numerics



(99|2|99)



(50|100|50)

Computational power

- Local manipulations (classical gates) project out edge states

  → loss of long ranged entanglement (LRE)

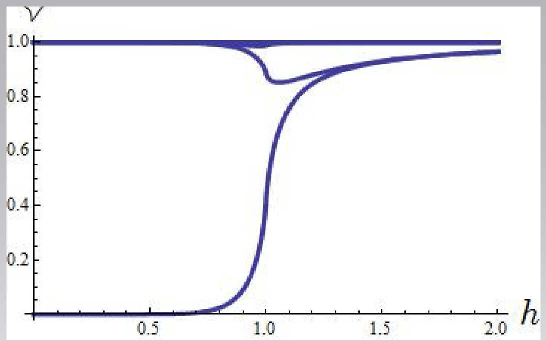
- A universal quantum simulator (UQS) must be endowed with LRE

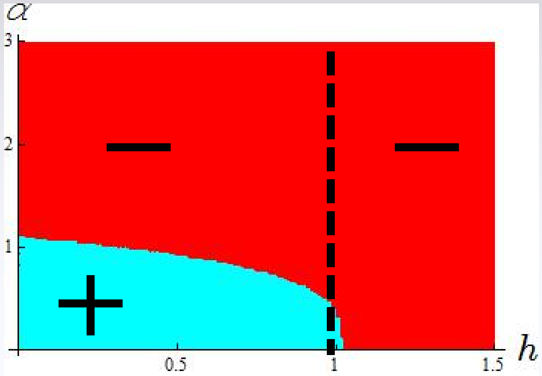
  ⇒ UQS must be non-locally convertible!

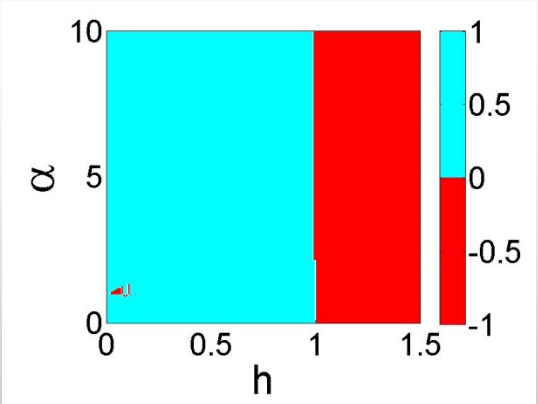
Conceivably: UQS could be connected to the Ising chain through an adiabatic path that does not cross a QPT

Conclusions

- Entanglement derivative to study non-local convertibility
- Local way of detecting long-range entanglement!
- Edge state recombination explains it
- Approaching a QPT:
  1. Correlation length increases
  2. Bulk states entanglement increases
  3. Edge states entanglement decreases
- Universal quantum simulators cannot be locally convertible







Thank you!

Quantum Computers & Simulators

- Certain problems too complex for classical computers: factorization, searches, simulation of quantum systems…
- Quantum algorithms give exponential speed-up, but implementation of quantum computers is hard
- Quantum systems as computers
  → Universal quantum simulator

Quantum Adiabatic Algorithm

- Ground state of H_I is the output of given problem
- Start from ground state of easy Hamiltonian H_0
- Adiabatically evolved it to desire state

$$H(t) = \left(1 - \frac{t}{T}\right) H_0 + \frac{t}{T} H_I$$

- If velocity sufficiently small ($T \ll \Delta_{\min}^{-2}$), system stays in instantaneous ground state

Computational power

- Any efficient quantum algorithm can be casted as a Quantum Adiabatic Algorithm
- Adiabatic evolution performs quantum computation
  $\rightarrow$ computational power of a quantum phase
- How to extract this computational power
  $\rightarrow$ Entanglement!

Entropy as a measure of entanglement

- Assume Bell State as unity of Entanglement:

$$|\text{Bell}\rangle = \frac{|\downarrow\downarrow\rangle \pm |\uparrow\uparrow\rangle}{\sqrt{2}}, \frac{|\downarrow\uparrow\rangle \pm |\downarrow\uparrow\rangle}{\sqrt{2}}$$

- Von Neumann Entropy measures how many Bell-Pairs can be distilled using LOCC from a given state $|\Psi^{A,B}\rangle$ (i.e. closeness of state to maximally entangled one)

What can entanglement entropy teach us about a system?

Entanglement

$$|0\rangle = \sum_{\kappa=1}^{2^L} \sqrt{\lambda_\kappa} |\Psi_\kappa^A\rangle \otimes |\Psi_\kappa^B\rangle \qquad S_\alpha = \frac{1}{1-\alpha} \log \sum_\kappa \lambda_\kappa^\alpha$$

- Quadratic Theory: Block eigenstates from block excitations

$$|\Psi_\kappa^A\rangle = |n_1, n_2, \ldots, n_L\rangle \ , \quad n_l = 0, 1$$

$$\lambda_\kappa = |\langle \Psi_\kappa^A | 0 \rangle|^2 = \prod_{j=1}^{L} \langle 0 | n_j \rangle \langle n_j | 0 \rangle$$

- Measure overlap of block excitaions with G.S.:

Whole system excitations: $c_j, c_j^\dagger \ \rightarrow c_j |0\rangle = 0$

Block excitation: $\tilde{c}_l, \tilde{c}_l^\dagger \rightarrow \tilde{c}_j |0\rangle \neq 0$

Entanglement

$$|0\rangle = \sum_{\kappa=1}^{2^L} \sqrt{\lambda_\kappa} |\Psi_\kappa^A\rangle \otimes |\Psi_\kappa^B\rangle \qquad S_\alpha = \frac{1}{1-\alpha} \log \sum_\kappa \lambda_\kappa^\alpha$$

- Block excitations from correaltion matrix:

Vidal & al, PRL (2003)

$$\langle f_k^{(a)} f_j^{(b)} \rangle = \delta_{j,k}\delta_{a,b} + i \left(\mathcal{B}_L\right)_{(j,k)}^{(a,b)} \xrightarrow{\text{eigenvalues}} \pm i\nu_j$$

$$\langle 0|0_j\rangle\langle 0_j|0\rangle = \langle 0|\tilde{c}_j \tilde{c}_j^\dagger|0\rangle = \frac{1+\nu_j}{2}$$

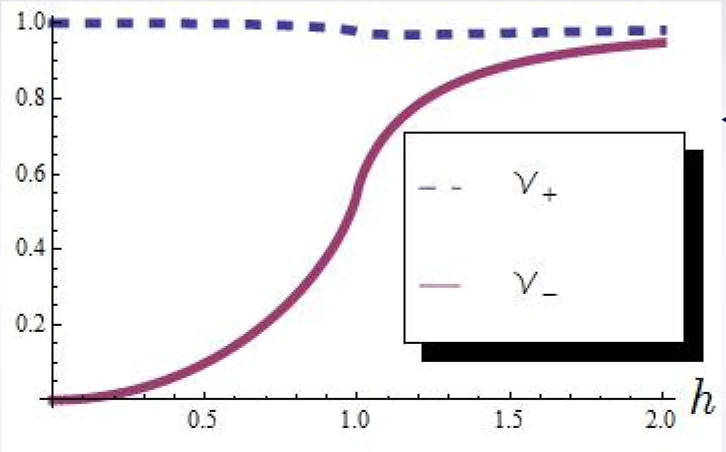
$$\langle 0|1_j\rangle\langle 1_j|0\rangle = \langle 0|\tilde{c}_j^\dagger \tilde{c}_j|0\rangle = \frac{1-\nu_j}{2}$$

Overlap between block excitations and ground state

$$\lambda_\kappa = \prod_{j=1}^{L} \langle 0|n_j\rangle\langle n_j|0\rangle = \prod_{j=1}^{L} \left(\frac{1 \pm \nu_j}{2}\right)$$

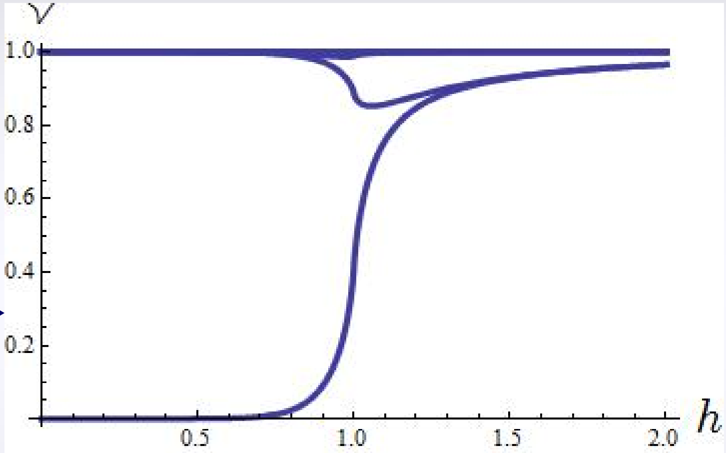
Correlation Matrix Eigenvalues

$$\langle f_k^{(a)} f_j^{(b)} \rangle = \delta_{j,k}\delta_{a,b} + i\,(\mathcal{B}_L)_{(j,k)}^{(a,b)} \longrightarrow \begin{cases} \langle 0|d_j d_j^\dagger|0\rangle = \dfrac{1+\nu_j}{2} \\ \langle 0|d_j^\dagger d_j|0\rangle = \dfrac{1-\nu_j}{2} \end{cases}$$



L=2

L=10

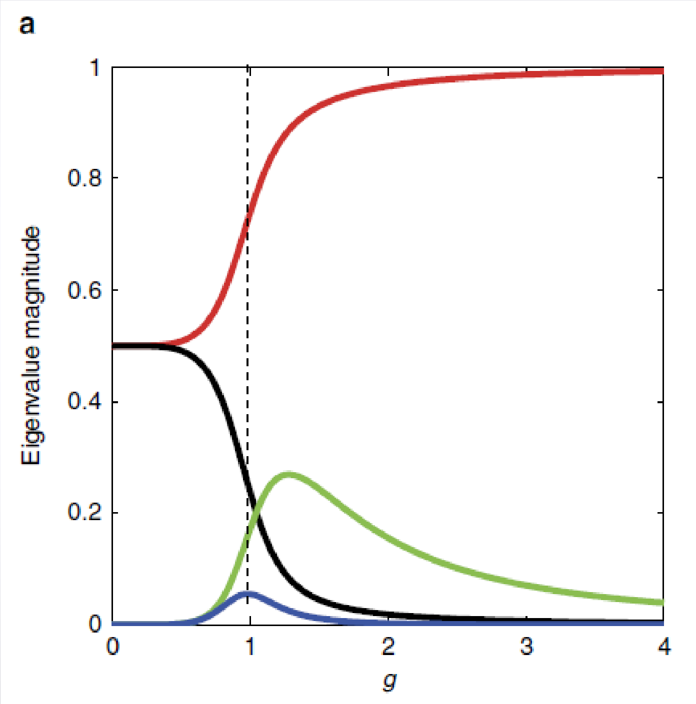


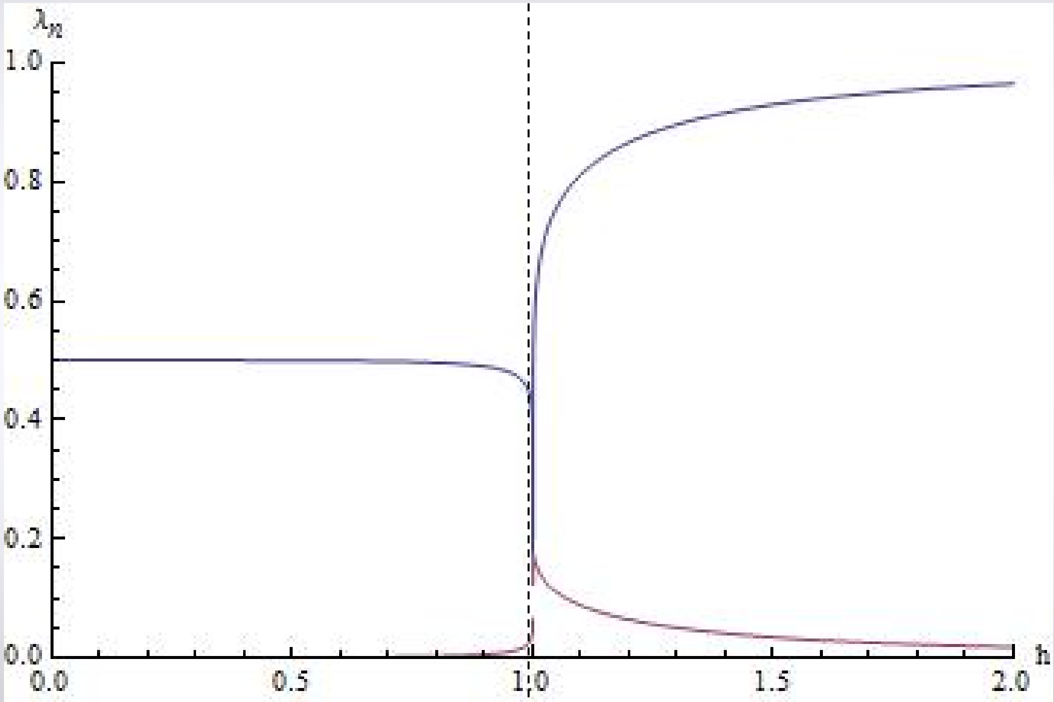
- One edge state for h<1: partial overlap
- Approaching QPT: bulk states overlap decreases, edge states overlap increases (edge state recombination)

Entanglement Spectrum

First few eigenvalues of the reduced density matrix

(multiplicities not shown)



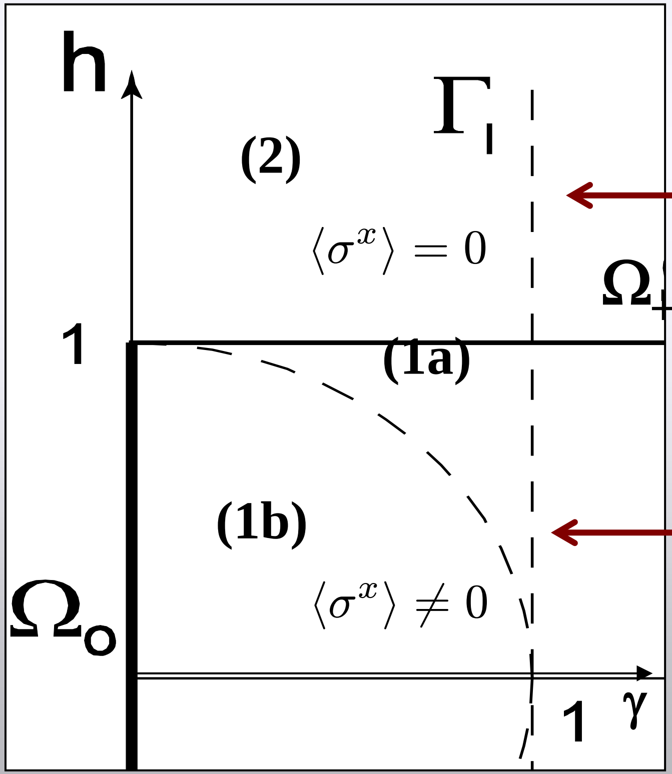


Finite Size
Numerical results

Theromodynamic Limit
Analytical results

Renyi Entropy

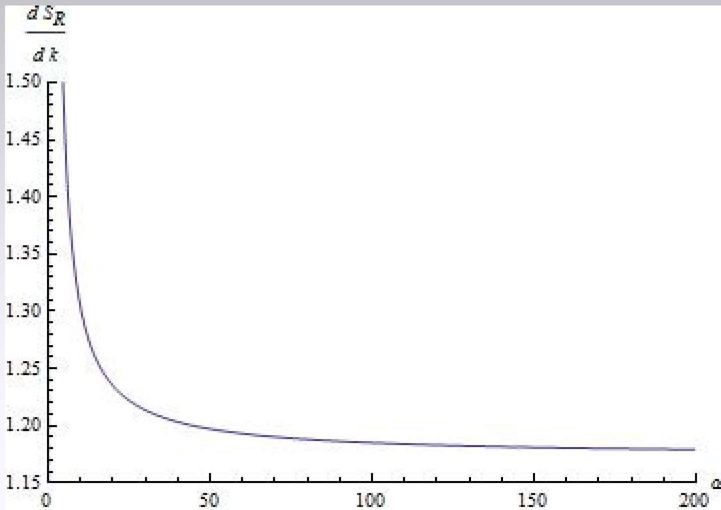
- Entropy depends on single parameter ε
- ε vanishes at phase transitions, large in gapped phase
- Microscopics of the model through ε(k)

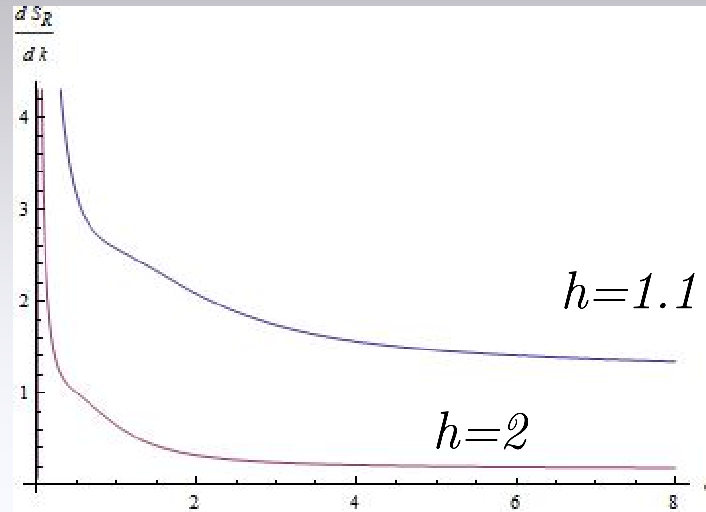


$$h > 1 \ : \ k = \frac{\gamma}{\sqrt{h^2+\gamma^2+1}} \rightarrow \frac{\mathrm{d}k}{\mathrm{d}h} < 0$$

$$h < 1 \ : \ k = \frac{\sqrt{h^2+\gamma^2+1}}{\gamma} \rightarrow \frac{\mathrm{d}k}{\mathrm{d}h} > 0$$

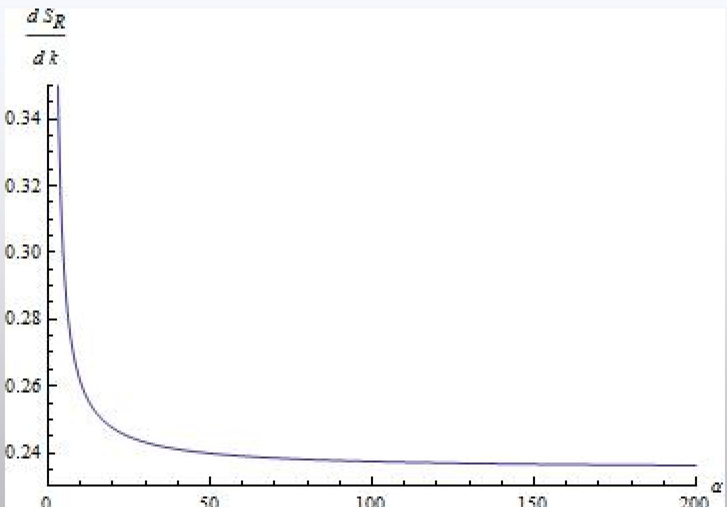
Entanglement Derivative

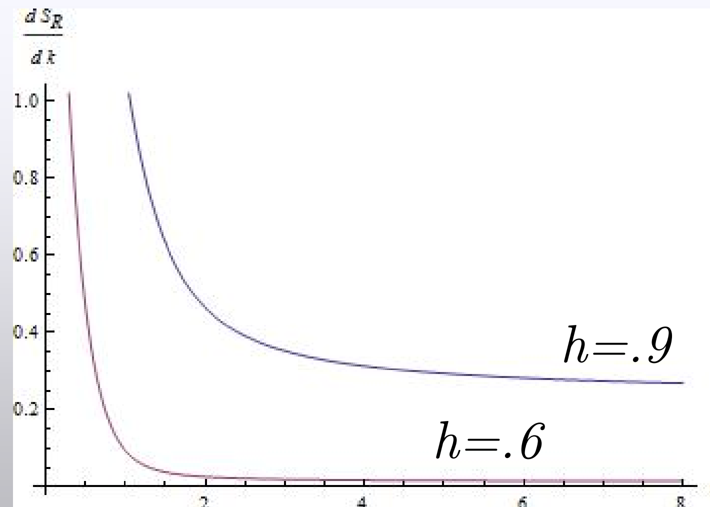




Paramagnetic Phase

$$\times \frac{\mathrm{d}k}{\mathrm{d}h} < 0$$





Ferromagnetic Phase

$$\times \frac{\mathrm{d}k}{\mathrm{d}h} > 0$$

L-spins subsystem

- Diagonalize L x L Hankel matrix:

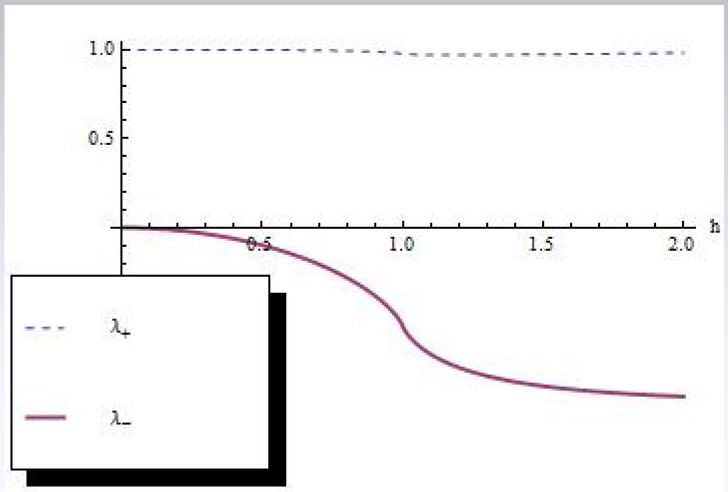
$$\tilde{\mathcal{B}} = \begin{pmatrix} g_{L-1} & g_{L-2} & \cdots & g_0 \\ g_{L-2} & g_{L-3} & \cdots & g_{-1} \\ \vdots & & \ddots & \vdots \\ g_0 & g_{-1} & \cdots & g_{1-L} \end{pmatrix}, \qquad g_j \equiv \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{ij\theta} \frac{\cos\theta - h + i\gamma \sin\theta}{\sqrt{(\cos\theta - h)^2 + \gamma^2 \sin^2\theta}} \, \mathrm{d}\theta$$

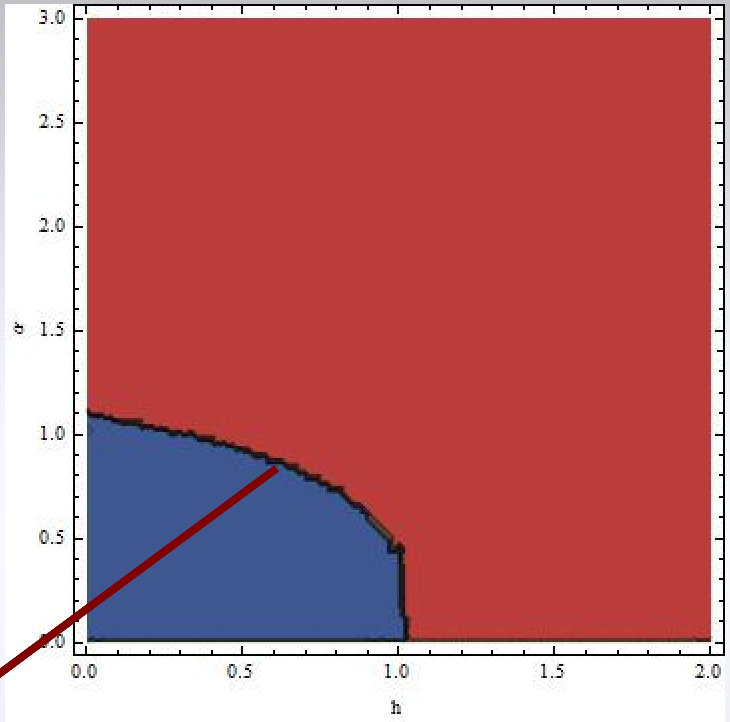
- Use L eigenvalues λ_j to compute Renyi entropy as sum of entropies of 2-levels systems:

$$S(\alpha) = \frac{1}{1-\alpha} \sum_{l=1}^{L} \ln \left[\left(\frac{1+\lambda_l}{2} \right)^{\alpha} + \left(\frac{1-\lambda_l}{2} \right)^{\alpha} \right]$$

$$\mathrm{d}S(\alpha) = \frac{\alpha}{1-\alpha} \sum_{l=1}^{L} \frac{(1+\lambda_l)^{\alpha-1} - (1-\lambda_l)^{\alpha-1}}{(1+\lambda_l)^{\alpha} + (1-\lambda_l)^{\alpha}} \, \mathrm{d}\lambda_l$$

2-spins subsystem: Ising line





- Entropy derivative vanishes in ferromagnetic phase!

  $\Rightarrow$ the two phases have different computation power!

- Role of Majorana edge states?